Lang

Introduction to Diophantine Approximations

New Expanded Edition

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Foreword

I thank Springer-Verlag for keeping my *Introduction to Diophantine Approximations* in print. This second edition is unchanged from the first, except for the addition of two papers, written in collaboration with W. Adams and H. Trotter, giving computational information for the behavior of certain algebraic and classical transcendental numbers with respect to approximation by rational numbers and their continued fractions. I thank both of them for their agreement to let me reproduce these papers, which expand and illustrate the general theory in computational directions.

The classical numbers, as I described them in 1965, are those which can be obtained by starting with the rational numbers, and performing the following operations:

– Take the algebraic closure, thus obtaining a field F.

– Take a classical, suitably normalized transcendental function (elliptic, hypergeometric, Bessel, exponential, logarithm, etc.), or jazzed up versions, coming from normalized transcendental parametrizations of algebraic varieties, take values of such functions with argument in F, and adjoint them to F.

– Iterate these two operations inductively.

Questions arise as to the properties of the numbers so obtained (a denumerable set), from the point of view of diophantine approximations. The present book may be viewed as providing the simplest examples at the most elementary level, using only the most elementary language of mathematics.

New Haven, 1995 SERGE LANG

Foreword to the First Edition

The quantitative aspects of the theory of diophantine approximations are, at the moment, still not very far from where Euler and Lagrange left them. Very recent work seems to have opened some fruitful lines of research, and in this book we shall illustrate by significant special examples three aspects from the theory of diophantine approximations.

First, the formal relationships which exist between various counting processes and functions entering in the theory. These essentially occur in Chapters I, II, III.

Second, the determination of these functions for numbers which are given as classical numbers, in a concrete fashion. Chapters IV and V give examples of this.

Third, we have mentioned certain asymptotic estimates holding almost everywhere (e.g. the Khintchine theorems and the Leveque–Erdös–Schmidt theorems). Such results are useful since they suggest roughly what may be considered "pathological" numbers, and also the range of magnitude of similar estimates for the classical numbers. However, as one sees from the quadratic numbers (which are of constant type), and the Adams result for e, each special number may exhibit its own particular behavior in the more subtle range of approximation. To determine this behavior for the classical numbers is perhaps the most fascinating part of the theory of diophantine approximations.

There exist other aspects, for instance the connection with transcendental numbers, but these have been left out completely since the style of the results known in this direction is at present so different from the style of the results which we have emphasized here.

I have avoided including partial results whose statements seemed to me too remote from expected best possible statements. Every chapter

should be viewed as working out a special case of a much broader general theory, as yet unknown. Indications for this are given throughout the book, together with references to current publications.

It is unusual to find a mathematical theory which is in a state as primitive and naive as the present one, and there is of course some delight in catching it in that state. In fact, this book may be used for a course in number theory, addressed to undergraduates, who will thus be put in contact with interesting but accessible problems on the ground floor of mathematics. If, however, like Rip van Winkle, I should awake from slumber in twenty years, my greatest hope would be that the theory by then had acquired the broad coherence which it deserves.

Berkeley, 1966 SERGE LANG

Contents

CHAPTER I

General Formalism

I, §1. RATIONAL CONTINUED FRACTIONS

We are interested in the following problem. Given an irrational number α, determine all solutions of the inequality

(1) $$|q\alpha - p| < \frac{1}{q} \quad \text{or} \quad \left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

with integers q, p, or more generally, of an inequality

(2) $$|q\alpha - p| < \psi(q),$$

where ψ is some positive decreasing function of a real variable. If ξ is a real number, we may write $\|\xi\|$ for the distance between ξ and the nearest integer. Then $|q\alpha - p| = \|q\alpha\|$ whenever it is sufficiently small, and we may rewrite the fundamental inequality (1) in the form $\|q\alpha\| < 1/q$. In investigating $\|q\alpha\|$, we are therefore interested only in the residue class of $q\alpha$ modulo $\mathbf{Z}$, where $\mathbf{Z}$ is the additive group of integers. The factor group $\mathbf{R}/\mathbf{Z}$ ($\mathbf{R}$ = real numbers) is sometimes called the circle group, since it is isomorphic to the group of complex numbers of absolute value 1 under the mapping $x \mapsto e^{2\pi i x}$. We may think of $\|\ \|$ as a metric on the circle $\mathbf{R}/\mathbf{Z}$.

The inequality (1) plays a fundamental role, and it turns out that one can describe most of its solutions by a rational process. In this chapter, we shall describe this process. The more general inequality (2) will be considered in the next chapter.

Unless otherwise specified, the results of this chapter are due to Euler and Lagrange. (For specific references, cf. Perron [21], which contains an extensive bibliography of the older literature, and is an excellent reference.)

We start by considering independent variables $a_0, a_1, a_2, \ldots$. We shall define inductively pairs of polynomials

$$p_n = p_n(a_0, \ldots, a_n) \quad \text{and} \quad q_n = q_n(a_0, \ldots, a_n),$$

starting with $p_0 = a_0$ and $q_0 = 1$. The quotient p_n/q_n will be written

$$\frac{p_n}{q_n} = [a_0, \ldots, a_n].$$

Suppose that we have defined p_k, q_k with $k < n$, and $n \geqq 1$. We shall use the abbreviation

$$p'_k = p_k(a_1, \ldots, a_n) \quad \text{and} \quad q'_k = q_k(a_1, \ldots, a_n).$$

We define inductively

$$p_n = a_0 p'_{n-1} + q'_{n-1} \quad \text{and} \quad q_n = p'_{n-1}. \tag{3}$$

This implies that

$$[a_0, \ldots, a_n] = \frac{p_n}{q_n} = a_0 + \frac{1}{[a_1, \ldots, a_n]}, \tag{4}$$

which, written in full, is equal to

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{}{\ddots \, a_{n-1} + \cfrac{1}{a_n}}}}}.$$

The sequence of fractions $\{p_n/q_n\}$ is called a **continued fraction**. When we substitute numbers for $a_0, a_1, \ldots$ such that q_n does not vanish, we obtain a sequence of numbers, which is still called a continued fraction.

Theorem 1. *We have for* $n \geqq 2$,

$$p_n = a_n p_{n-1} + p_{n-2},$$

$$q_n = a_n q_{n-1} + q_{n-2}.$$

Proof. For $n = 2$, the assertion is verified directly. Assume $n > 2$, and assume inductively that

$$p'_{n-1} = a_n p'_{n-2} + p'_{n-3},$$
$$q'_{n-1} = a_n q'_{n-2} + q'_{n-3},$$

Using (3), we find

$$\begin{aligned} p_n &= a_0(a_n p'_{n-2} + p'_{n-3}) + a_n q'_{n-2} + q'_{n-3} \\ &= a_n(a_0 p'_{n-2} + q'_{n-2}) + a_0 p'_{n-3} + q'_{n-3} \\ &= a_n p_{n-1} + p_{n-2} \end{aligned}$$

and

$$q_n = a_n p'_{n-2} + p'_{n-3} = a_n q_{n-1} + q_{n-2},$$

thereby proving our theorem.

For convenience, we define $p_{-1} = 1$ and $q_{-1} = 0$. Then Theorem 1 remains valid when $n = 1$.

In applications, we shall be interested in the nature of the values of p_n and q_n when we substitute real numbers, for $a_0, a_1, \ldots$. We shall always take $a_1, a_2, \ldots$ to be > 0. In that case, we see inductively that $q_n > 0$ for $n \geqq 1$, and hence that the fraction p_n/q_n has meaning as a real number. In particular, we obtain a corollary for Theorem 1.

Corollary 1. *Let a_i be real numbers > 0 for $1 \leqq i \leqq n$. For $1 \leqq k \leqq n$, let*

$$r_k = [a_k, \ldots, a_n].$$

Then

$$\begin{aligned} [a_0, \ldots, a_n] &= [a_0, \ldots, a_{k-1}, r_k] \\ &= \frac{p_{k-1} r_k + p_{k-2}}{q_{k-1} r_k + q_{k-2}}. \end{aligned}$$

Corollary 2. *Let $a_0, \ldots, a_n$ and $b_0, \ldots, b_n$ be real numbers such that $a_i \geqq 1$ for $i \geqq 1$ and also $b_i \geqq 1$ for $i \geqq 1$. Assume that a_j, b_j are integers for $0 \leqq j \leqq n - 1$. If*

$$[a_0, \ldots, a_n] = [b_0, \ldots, b_n],$$

then $a_i = b_i$ for $i \geqq 0$.

Proof. Let $r_1 = [a_1, \ldots, a_n]$, so that

$$r_1 = a_1 + \frac{1}{[a_2, \ldots, a_n]}.$$

Then $r_1 \geqq 1$, and similarly, $s_1 = [b_1, \ldots, b_n] \geqq 1$. By hypothesis,

$$a_0 + \frac{1}{r_1} = b_0 + \frac{1}{s_1}.$$

If $r_1 = 1$, then

$$a_0 + \frac{1}{r_1}$$

is an integer, and hence $1/s_1$ is also an integer. Hence $s_1 = 1$ and $a_0 = b_0$. If $r_1 > 1$, then

$$a_0 + \frac{1}{r_1}$$

is not an integer, and hence $s_1 > 1$ also. But then $a_0 = b_0$ because both a_0, b_0 are the greatest integers $\leqq [a_0, \ldots, a_n]$. Thus in all cases, $a_0 = b_0$ and $r_1 = s_1$. We can now conclude the proof by induction.

Corollary 2 gives us a uniqueness for the continued fraction formed with real numbers under the hypotheses of this corollary. It will be applicable in the next section.

For the next theorem, we return to indeterminates.

Theorem 2. *For $n \geqq 0$ we have*

$$q_n p_{n-1} - p_n q_{n-1} = (-1)^n.$$

Proof. We have

$$q_0 p_{-1} - p_0 q_{-1} = 1.$$

In general, multiply the first expression in Theorem 1 by q_{n-1}, multiply the second by p_{n-1}, and subtract the first from the second. We obtain

$$q_n p_{n-1} - p_n q_{n-1} = -(q_{n-1} p_{n-2} - p_{n-1} q_{n-2}).$$

This proves the theorem, because it shows that when n changes by one unit, the expression on the left of the inequality in the theorem changes by a minus sign.

Corollary 1. *For $n \geqq 1$ we have*

$$\frac{p_{n-1}}{q_{n-1}} - \frac{p_n}{q_n} = \frac{(-1)^n}{q_n q_{n-1}}.$$

We shall be interested in the values taken by p_n and q_n when $a_0, a_1, \ldots$ are integers. We shall assume throughout that when we substitute such integers, then $a_1, a_2, \ldots$ are always > 0.

Corollary 2. *If $a_1, a_2, \ldots$ are positive integers, then p_n and q_n are relatively prime, and*

$$0 < q_1 < q_2 < \cdots$$

forms a strictly increasing sequence of integers.

Corollary 3. *Let α denote the rational function*

$$\alpha = [a_0, \ldots, a_{n+2}].$$

Then

$$q_{n+1}\alpha - p_{n+1} = \frac{(-1)^{n+1}}{a_{n+2}q_{n+1} + q_n}.$$

Proof. Replace n by $n + 2$ in the theorem, and divide by q_{n+2}. Note that by definition, $p_{n+2}/q_{n+2} = \alpha$, and $q_{n+2} = a_{n+2}q_{n+1} + q_n$. The relation of our corollary then drops out.

Theorem 3. *For $n \geqq 1$ we have*

$$q_n p_{n-2} - p_n q_{n-2} = (-1)^{n-1} a_n.$$

Proof. We multiply the first expression in Theorem 1 by q_{n-2}, multiply the second by p_{n-2} and subtract the first from the second. We obtain, using Theorem 2,

$$q_n p_{n-2} - p_n q_{n-2} = a_n(q_{n-1}p_{n-2} - p_{n-1}q_{n-2}) = (-1)^{n-1}a_n,$$

as was to be shown.

Corollary 1. *For $n \geqq 2$ we have*

$$\frac{p_{n-2}}{q_{n-2}} - \frac{p_n}{q_n} = \frac{(-1)^{n-1}a_n}{q_n q_{n-2}}.$$

Corollary 2. *If $a_1, a_2, \ldots$ are positive numbers, then the sequence p_n/q_n for even n is strictly increasing, and for odd n, it is strictly decreasing.*

Corollary 3. *Let α denote the rational function*

$$\alpha = [a_0, \ldots, a_{n+2}].$$

Then

$$q_n\alpha - p_n = \frac{(-1)^n a_{n+2}}{a_{n+2}q_{n+1} + q_n}.$$

Proof. Replace n by $n+2$ in Theorem 3, and divide by q_{n+2}. Note that by definition, $p_{n+2}/q_{n+2} = \alpha$, and $q_{n+2} = a_{n+2}q_{n+1} + q_n$. The relation of our corollary then drops out.

Theorem 4. *For $n \geqq 1$ we have*

$$\frac{q_n}{q_{n-1}} = [a_n, a_{n-1}, \ldots, a_1].$$

Proof. For $n = 1$, the assertion is clear. Assume $n > 1$. Suppose inductively that we know

$$\frac{q_{n-1}}{q_{n-2}} = [a_{n-1}, \ldots, a_1].$$

Since $q_n = a_n q_{n-1} + q_{n-2}$ we find

$$\frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-2}}{q_{n-1}} = a_n + \frac{1}{[a_{n-1}, \ldots, a_1]},$$

and our assertion follows by induction.

I, §2. THE CONTINUED FRACTION OF A REAL NUMBER

Consider first briefly the special case of a *rational number* α. Let a_0 be the largest integer $\leqq \alpha$. If α is not an integer, we can write

$$\alpha = a_0 + \frac{1}{\alpha_1}$$

with $\alpha_1 > 1$, and α_1 is again rational. Inductively, we let

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}},$$

where a_n is the largest integer $\leqq \alpha_n$ and $\alpha_{n+1} > 1$. We can do this provided α_n is not an integer. However, our process will stop after a finite number of steps. Indeed, if $\alpha_n = a/b$ is a rational number, not an integer, with positive integers a, b, then

$$\alpha_n - a_n = \frac{a - ba_n}{b} = \frac{c}{b}$$

with $c < b$. Then

$$\alpha_{n+1} = \frac{b}{c}$$

and hence the denominator of α_{n+1} is smaller than the denominator of α. So the process stops, and we can write our rational number α in the form

$$\alpha = [a_0, a_1, \ldots, a_n],$$

with integers a_i $(i = 0, \ldots, n)$, and $a_i \geqq 1$ for $i \geqq 1$. Observe that we have a choice for the last partial quotient a_n, namely we can write α in the above form with a_n equal to an integer > 1, or also in the form

$$\alpha = [a_0, a_1, \ldots, a_n - 1, 1].$$

Thus the length of the continued fraction expansion of a rational number may be taken to be either even or odd.

Let α be a real irrational number. We can determine a continued fraction for α by writing as before $\alpha = a_0 + 1/\alpha_1$ with a_0 equal to the largest integer $\leqq \alpha$ (that is, $a_0 = [\alpha]$), and $\alpha_1 > 1$. Inductively, we let

$$\alpha_n = a_n + \frac{1}{\alpha_{n+1}},$$

where a_n is the largest integer $\leqq \alpha_n$, and $\alpha_{n+1} > 1$. Since α is irrational, the sequence $\alpha_1, \alpha_2, \ldots$ does not terminate. We have, in the notation of §1,

$$\alpha = [a_0, a_1, \ldots, a_n, \alpha_{n+1}]$$

for any $n \geqq 0$. We shall also write symbolically the infinite expression

$$\alpha = [a_0, a_1, \ldots].$$

From Corollary 2 of Theorem 2, we obtain a sequence of relatively prime integers p_n, q_n with $q_n \geqq 1$, belonging to the continued fraction $[a_0, \ldots, a_n]$, and thus the relation

$$\frac{p_n}{q_n} = [a_0, \ldots, a_n]$$

is now a relation between real numbers, not any more between indeterminates. Furthermore, p_n/q_n is a reduced fraction, which will be called the n-th **principal convergent** of α. We call a_n the n-th **partial quotient** of α.

The formalism of §1 now applies to the continued fraction for α. For instance, the Corollaries 3 of Theorems 2 and 3 must now be written

$$\boxed{q_{n+1}\alpha - p_{n+1} = \frac{(-1)^{n+1}}{\alpha_{n+2}q_{n+1} + q_n}}$$

and

$$\boxed{q_n\alpha - p_n = \frac{(-1)^n\alpha_{n+2}}{\alpha_{n+2}q_{n+1} + q_n}}$$

We always have

$$a_n < \alpha_n < a_n + 1,$$

and $a_n \geqq 1$ for all $n \geqq 1$. Hence the denominators q_n are all positive integers, and form an increasing sequence,

$$0 < q_1 < \cdots < q_n < q_{n+1} < \cdots.$$

Theorem 5. *For even* n, *the* n-th *principal convergents of* α *form a strictly increasing sequence converging to* α. *For odd* n, *the* n-th *principal convergents of* α *form a strictly decreasing sequence converging to* α. *Furthermore, we have*

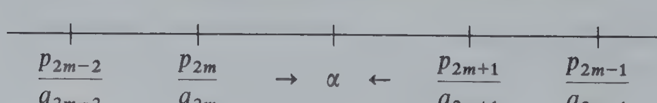
$$\frac{1}{2q_{n+1}} < \frac{1}{q_{n+1} + q_n} < |q_n\alpha - p_n| < \frac{1}{q_{n+1}}.$$

Proof. The first assertion follows from Corollary 2 of Theorem 3, §1, and Corollary 1 of Theorem 2. So does the inequality on the right. The left inequality follows from Corollary 1 of Theorem 3, §1, namely

$$\left|\alpha - \frac{p_n}{q_n}\right| > \left|\frac{p_{n+2}}{q_{n+2}} - \frac{p_n}{q_n}\right| = \frac{a_{n+2}}{q_{n+2}q_n} = \frac{a_{n+2}}{(a_{n+2}q_{n+1} + q_n)q_n}.$$

We divide numerator and denominator by a_{n+2} and use the fact that $a_{n+2} \geqq 1$ to conclude the proof.

The picture illustrating Theorem 5 may be drawn as follows:



Since $q_{n+1} > q_n$, we find that our convergents give us solutions of the inequality (1), namely

$$|q_n\alpha - p_n| < \frac{1}{q_n}.$$

We shall determine in §3 what other possible solutions may exist.

We observe that for $n \geqq 1$ we have

$$|q_n\alpha - p_n| = \|q_n\alpha\|.$$

Corollary. *For $n \geqq 2$ we have*

$$\|q_{n-1}\alpha\| = a_n\|q_n\alpha\| + \|q_{n+1}\alpha\|,$$

whence

$$\|q_n\alpha\| < \|q_{n-1}\alpha\|,$$

and

$$a_n = \left[\frac{\|q_{n-1}\alpha\|}{\|q_n\alpha\|}\right].$$

Proof. We use Theorem 1 to express $q_{n+1}\alpha - p_{n+1}$, and then use the fact that $q_{n+1}\alpha - p_{n+1}$ and $q_n\alpha - p_n$ have opposite signs by Theorem 5. This proves the first relation of the corollary. The others are immediate consequences.

We shall now characterize the principal convergents to α by an ordering property.

A **best approximation** to α is a fraction p/q $(q > 0)$ such that

$$\|q\alpha\| = |q\alpha - p|, \qquad \text{and} \qquad \|q'\alpha\| > \|q\alpha\|$$

if $1 \leqq q' < q$. Observe that the fraction p/q is necessarily reduced (i.e. p, q must be relatively prime) if it is a best approximation to α, for otherwise, we can write $p = p'r$, $q = q'r$ with $r > 1$, and $q' < q$, so that

$$|q'\alpha - p'| < |q\alpha - p|,$$

which is impossible.

Theorem 6. *The best approximations to α are the principal convergents to α. In fact, for $n \geqq 1$, q_n is the smallest integer $q > q_{n-1}$ such that $\|q\alpha\| < \|q_{n-1}\alpha\|$.*

Proof. Let us first show that a best approximation is a convergent. Let a/b be a reduced fraction, $b > 0$, which is a best approximation to α. We must show that $a/b = p_n/q_n$ for some n. Suppose that $a/b < p_0/q_0 = a_0$. Then

$$|\alpha - a_0| < \left|\alpha - \frac{a}{b}\right| \leqq |b\alpha - a|,$$

contradicting the hypothesis. Suppose that $a/b > p_1/q_1$. Then

$$\left|\frac{a}{b} - \alpha\right| > \left|\frac{a}{b} - \frac{p_1}{q_1}\right| \geqq \frac{1}{bq_1},$$

whence

$$|b\alpha - a| > \frac{1}{q_1} = \frac{1}{a_1} \geqq |\alpha - a_0|,$$

again contradicting the hypothesis. Finally, suppose that a/b lies between p_{n-1}/q_{n-1} and p_{n+1}/q_{n+1}, but is not equal to either of these fractions. Then

$$\frac{1}{bq_{n-1}} \leqq \left|\frac{a}{b} - \frac{p_{n-1}}{q_{n-1}}\right| < \left|\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}}\right| = \frac{1}{q_n q_{n-1}}.$$

Hence $q_n < b$. On the other hand,

$$\frac{1}{bq_{n+1}} \leqq \left|\frac{p_{n+1}}{q_{n+1}} - \frac{a}{b}\right| \leqq \left|\alpha - \frac{a}{b}\right|$$

whence

$$|q_n\alpha - p_n| < \frac{1}{q_{n+1}} \leqq |b\alpha - a|,$$

contradiction. This proves the first half of the theorem.

We shall prove the converse, by induction on n. First for $n = 0$, since $q_0 = 1$, there is no q such that $1 \leqq q < q_0$. Hence the definition of best approximation is vacuously satisfied by p_0/q_0. Assume now that our assertion has been proved for p_n/q_n with $n \geqq 0$. We wish to prove that p_{n+1}/q_{n+1} is a best approximation. Let q be the smallest integer $> q_n$ such that

$$\|q\alpha\| < \|q_n\alpha\|,$$

and let p be such that $\|q\alpha\| = |q\alpha - p|$. Then by the inductive property that p_n/q_n is a best approximation, we conclude that p/q is a best approximation also, and hence must be a principal convergent by what has already been shown. Since q is chosen smallest $> q_n$ such that $\|q\alpha\| < \|q_n\alpha\|$, it follows that $q = q_{n+1}$. But then $p = p_{n+1}$ (trivially), thereby proving our theorem.

Corollary 1. *If p/q is a principal convergent to α, and m is an integer with $1 \leqq m < q$, then*

$$\frac{1}{2q} < \|m\alpha\|.$$

Proof. Suppose that $q = q_n$. By Theorem 5, we have

$$\frac{1}{2q_n} < \|q_{n-1}\alpha\|,$$

and by Theorem 6, $\|q_{n-1}\alpha\| \leqq \|m\alpha\|$, as was to be shown.

Corollary 2. *If a/b is a reduced fraction, $b > 0$, such that*

$$\left|\alpha - \frac{a}{b}\right| < \frac{1}{2b^2},$$

then a/b is a principal convergent to α.

Proof. By the theorem, it will suffice to prove that a/b is a best approximation to α. Let c/d be a fraction, $d > 0$, $c/d \neq a/b$, such that

$$|d\alpha - c| \leqq |b\alpha - a| < \frac{1}{2b}.$$

Then

$$\frac{1}{bd} \leqq \left|\frac{c}{d} - \frac{a}{b}\right| \leqq \left|\alpha - \frac{c}{d}\right| + \left|\alpha - \frac{a}{b}\right| < \frac{1}{2bd} + \frac{1}{2b^2} = \frac{b+d}{2b^2d}.$$

From this we conclude $b < d$, whence a/b is a best approximation to α, as was to be shown.

I, §3. EQUIVALENT NUMBERS

The set of matrices

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

with integral components a, b, c, d having determinant ± 1 (i.e. $ad - bc = 1$ or -1) is in fact a group, for the product of two such matrices and the inverse of such a matrix again have determinant ± 1. Let G be this group. If α is an irrational number, and σ is as above an element of G, we define

$$\sigma\alpha = \frac{a\alpha + b}{c\alpha + d}.$$

Then one verifies by brute force that if $\sigma, \tau \in G$ then

$$\sigma(\tau\alpha) = (\sigma\tau)\alpha \qquad \text{and} \qquad I\alpha = \alpha$$

if I denotes the unit 2×2 matrix. Thus G operates on the set of irrational numbers, and we shall say that two irrational numbers α, β are **equivalent** if there exists $\sigma \in G$ such that $\sigma\alpha = \beta$. It is trivially verified that this is an equivalence relation.

Example 1. For α irrational, we can write

$$\alpha = [a_0, a_1, \ldots, a_{n-1}, \alpha_n].$$

By Theorem 1 of §1, and its corollary, we obtain for $n \geqq 1$,

$$\alpha = \frac{p_{n-1}\alpha_n + p_{n-2}}{q_{n-1}\alpha_n + q_{n-2}}.$$

Let

$$\sigma_{n-1} = \begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix}$$

and call σ_{n-1} the $(n-1)$-th **continued transformation** of α. We see that

$$\alpha = \sigma_{n-1}\alpha_n.$$

Furthermore, Theorem 2 of §1 shows that σ_{n-1} is an element of our group G. Thus α is equivalent to α_n for $n \geqq 1$, and consequently all numbers α_n $(n = 1, 2, \ldots)$ are equivalent to each other. We also note that if we let

$$A_n = \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix}$$

then $\det A_n = -1$, and from Theorem 1 of §1, we find

$$\sigma_n = A_0 \cdots A_n.$$

In the next theorem, we give a characterization of the situation described in our example.

Theorem 7. *Let α, β be irrational, and*

$$\sigma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G, \qquad \alpha = \sigma\beta = \frac{a\beta + b}{c\beta + d}.$$

Assume that $\beta > 1$, and $c > d > 0$. Then b/d and a/c are two successive principal convergents to α, say p_{n-2}/q_{n-2} and p_{n-1}/q_{n-1}, and $\beta = \alpha_n$.

Proof. Note that a, c are relatively prime because $ad - bc = \pm 1$. We can express a/c as a continued fraction,

$$\frac{a}{c} = [a_0, \ldots, a_{n-1}] = \frac{p_{n-1}}{q_{n-1}},$$

and we have $a = p_{n-1}$, $c = q_{n-1}$. We may choose n so that

$$p_{n-1}q_{n-2} - q_{n-1}p_{n-2} = \epsilon,$$

where $\epsilon = ad - bc$ (because the continued fraction of a rational number can be shortened or lengthened by 1 artificially). Since

$$ad - bc = p_{n-1}d - q_{n-1}b = \epsilon$$

we find

$$p_{n-1}(d - q_{n-2}) = q_{n-1}(b - p_{n-2}).$$

Since p_{n-1}, q_{n-1} are relatively prime, it follows that q_{n-1} divides $(d - q_{n-2})$. But $q_{n-2} \leqq q_{n-1}$ and $d < q_{n-1}$. Hence

$$|d - q_{n-2}| < q_{n-1},$$

and therefore $d - q_{n-2} = 0$. Then $b - p_{n-2} = 0$. Thus we can write

$$\alpha = \frac{p_{n-1}\beta + p_{n-2}}{q_{n-1}\beta + q_{n-2}}.$$

This means that

$$\alpha = [a_0, \ldots, a_{n-1}, \beta].$$

Since $\beta > 1$, it follows that the above expression is the continued fraction expansion of α, and that $\beta = \alpha_n$. We then see finally that a/c and b/d are consecutive principal convergents, as was to be shown.

Theorem 8 (Serret). *Let α, β be irrational numbers. They are equivalent if and only if $\alpha_n = \beta_m$ for some pair of integers n, $m \geqq 1$, or equivalently, in their continued fractions*

$$\alpha = [a_0, a_1, a_2, \ldots],$$
$$\beta = [b_0, b_1, b_2, \ldots],$$

we have $a_n = b_{n+l}$ for some l and all n sufficiently large.

Proof. Assume that there exist integers $k, l \geqq 1$ such that $\alpha_k = \beta_l$, i.e.

$$\alpha = [a_0, a_1, \ldots, a_{k-1}, \alpha_k],$$
$$\beta = [b_0, b_1, \ldots, b_{l-1}, \beta_l],$$

and $\alpha_k = \beta_l$. Since we have seen that α is equivalent to α_k, and β is equivalent to β_l, it follows that α is equivalent to β.

Conversely, assume that α, β are equivalent, say

$$\beta = \frac{a\alpha + b}{c\alpha + d} = \sigma\alpha$$

with $ad - bc = \pm 1$. Without loss of generality, we may assume that $c\alpha + d > 0$ (otherwise, replace a, b, c, d by their negatives). Let σ_{n-1} be as in Example 1, so that $\alpha = \sigma_{n-1}\alpha_n$. Then

$$\beta = \sigma\sigma_{n-1}\alpha_n,$$

and

$$\sigma\sigma_{n-1} = \begin{pmatrix} * & * \\ cp_{n-1} + dq_{n-1} & cp_{n-2} + dq_{n-2} \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}.$$

We have:

$$cp_{n-1} + dq_{n-1} = q_{n-1}\left(c\frac{p_{n-1}}{q_{n-1}} + d\right) = c',$$

$$cp_{n-2} + dq_{n-2} = q_{n-2}\left(c\frac{p_{n-2}}{q_{n-2}} + d\right) = d'.$$

We take n large, so that p_{n-1}/q_{n-1} and p_{n-2}/q_{n-2} are close to α. Then c' and d' are both > 0, and also $\alpha_n > 1$. Finally, we can take the parity of n so that $c' > d'$. Then all the conditions of Theorem 7 are satisfied, and we conclude that $\alpha_n = \beta_m$ for some m. This proves our theorem.

Examples. Suppose first that

$$\alpha = [a_0, a_1, a_2, \ldots].$$

Then it is easily verified that

$$-\alpha = \begin{cases} [-a_0 - 1, 1, a_1 - 1, a_2, a_3, \ldots] & \text{if } a_1 > 1, \\ [-a_0 - 1, a_2 + 1, a_3, a_4, \ldots] & \text{if } a_1 = 1. \end{cases}$$

It is also easy to take the inverse. We have from the definitions:

$$1/\alpha = \begin{cases} [0, a_0, a_1, \ldots] & \text{if } \alpha > 1, \\ [a_1, a_2, a_3, \ldots] & \text{if } 0 < \alpha < 1. \end{cases}$$

I, §4. INTERMEDIATE CONVERGENTS

We continue with more results of Lagrange, and define fractions lying between the principal convergents of an irrational number α. We first note that if a, b, c, d are non-zero real numbers such that

$$\frac{a}{b} < \frac{c}{d}.$$

and r is a positive number, with $b + rd \neq 0$, then

$$\frac{a}{b} < \frac{a + rc}{b + rd} < \frac{c}{d}.$$

The proof is trivial by cross multiplying. Furthermore, if $r < s$ then

$$\frac{a + rc}{b + rd} < \frac{a + sc}{b + sd},$$

in other words, the quotient $(a + rc)/(b + rd)$ is a strictly increasing function of r. For any integer $r \geqq 0$ we let

$$p_{n,r} = rp_{n+1} + p_n \qquad \text{and} \qquad q_{n,r} = rq_{n+1} + q_n.$$

If $r = 0$, then $p_{n,0} = p_n$, and if $r = a_{n+2}$ then $p_{n,r} = p_{n+2}$. Similarly for $q_{n,r}$. We shall be interested in the values of r such that

$$1 \leqq r < a_{n+2},$$

and call the fractions

$$\frac{p_{n,r}}{q_{n,r}} = \frac{rp_{n+1} + p_n}{rq_{n+1} + q_n}, \qquad 1 \leqq r \leqq a_{n+2} - 1,$$

the n-th **intermediate convergents** of the continued fraction $[a_0, a_1, \ldots]$, or of α if this continued fraction is the one associated with α. An intermediate convergent or a principal convergent will be called a **convergent**.

We note that the denominators of the intermediate convergents and the convergents form a strictly increasing sequence

$$\cdots < q_{n+1} < \cdots < q_{n,r} < q_{n,r+1} < \cdots < q_{n+2} < \cdots.$$

Theorem 9. *For n even, we have a strictly increasing sequence*

$$\cdots < \frac{p_n}{q_n} < \cdots < \frac{p_{n,r}}{q_{n,r}} < \frac{p_{n,r+1}}{q_{n,r+1}} < \cdots < \frac{p_{n+2}}{q_{n+2}} < \cdots$$

and a similar decreasing sequence for n odd. Furthermore,

$$q_{n,r+1}p_{n,r} - p_{n,r+1}q_{n,r} = (-1)^{n+1}.$$

Proof. The increasing sequence comes from the remarks on inequalities made at the beginning of the section. The last relation follows from a trivial computation, using Theorem 2, §1.

We conclude that $p_{n,r}$ and $q_{n,r}$ are relatively prime, and thus the intermediate convergent $p_{n,r}/q_{n,r}$ is in reduced form.

The next result, which is important for the determination of the solutions of the fundamental inequality $|q\alpha - p| < 1/q$ is due to Grace [10] (cf. also Adams [1]).

Theorem 10. *If p, q are non-zero integers, $q > 0$, satisfying the inequality $|\alpha - p/q| < 1/q^2$, then p/q is a convergent of α, and is in fact equal to some $p_{n,r}/q_{n,r}$ with $r = 0$, or $r = 1$, or $r = a_{n+2} - 1$.*

Proof. For the first statement, assume, say that $\alpha < p/q$, the other case being proved in a similar manner. If p/q is not a convergent, then there exist two successive convergents P/Q and P'/Q', such that

$$\alpha < P/Q < p/q < P'/Q'$$

and $P'Q - PQ' = 1$, by Theorem 9. Thus

$$\frac{1}{q^2} > \frac{p}{q} - \alpha > \frac{p}{q} - \frac{P}{Q} \geqq \frac{1}{qQ}$$

and

$$\frac{1}{Q'q} \leqq \frac{P'}{Q'} - \frac{p}{q} < \frac{P'}{Q'} - \frac{P}{Q} = \frac{1}{Q'Q}.$$

These estimates are contradictory, and our first assertion is proved.

Conversely, we must determine which convergents $p_{n,r}/q_{n,r}$ satisfy the desired inequality. We need a lemma.

Lemma. *If $p_{n,r}/q_{n,r}$ is an intermediate convergent of α, then*

$$q_{n,r}\alpha - p_{n,r} = \frac{(-1)^n(\alpha_{n+2} - r)}{\alpha_{n+2}q_{n+1} + q_n}.$$

Proof. We have the continued fraction

$$\alpha = [a_0, a_1, \ldots, a_{n+1}, \alpha_{n+2}].$$

The relation of Corollary 3 of Theorem 3, §1, is a relation between indeterminates. Consequently it holds for the special values of the preceding continued fraction, and we find

$$q_n\alpha - p_n = \frac{(-1)^n\alpha_{n+2}}{\alpha_{n+2}q_{n+1} + q_n}.$$

Similarly, using Corollary 3 of Theorem 2, §1, we find

$$q_{n+1}\alpha - p_{n+1} = \frac{(-1)^{n+1}}{\alpha_{n+2}q_{n+1} + q_n}.$$

Multiplying this last expression by r and adding the preceding one, we find the equality stated in the assertion of the lemma.

We observe that $r < \alpha_{n+2}$ because $r \leqq a_{n+2} - 1$, and consequently that

$$|q_{n,r}\alpha - p_{n,r}| = \frac{\alpha_{n+2} - r}{\alpha_{n+2}q_{n+1} + q_n}.$$

We must find a necessary condition on r for this expression to be $< 1/q_{n,r}$, in other words,

$$\frac{\alpha_{n+2} - r}{\alpha_{n+2}q_{n+1} + q_n} < \frac{1}{rq_{n+1} + q_n}.$$

A trivial manipulation shows that this inequality is equivalent to

$$\frac{\left(q_{n+1} + \dfrac{q_n}{r}\right)}{\left(q_{n+1} + \dfrac{q_n}{\alpha_{n+2}}\right)} r\left(1 - \frac{r}{\alpha_{n+2}}\right) < 1.$$

Suppose $r > 0$. Since $r < \alpha_{n+2}$, this inequality implies

$$r\left(1 - \frac{r}{\alpha_{n+2}}\right) < 1.$$

Suppose $r < a_{n+2} - 1$ and hence $r < \alpha_{n+2} - 2$ because r is an integer and

α_{n+2} is not. Then

$$r\left(1 - \frac{\alpha_{n+2} - 2}{\alpha_{n+2}}\right) < 1,$$

whence $r < \alpha_{n+2}/2$. Consequently

$$r\left(1 - \frac{\alpha_{n+2}}{2\alpha_{n+2}}\right) < 1,$$

and $r < 2$, so that $r = 1$. This proves Theorem 10.

We can generalize to intermediate convergents one of the inequalities proved previously for the principal convergents of α.

Theorem 11. *Let $p_{n,r}/q_{n,r}$ be a convergent of α with*

$$0 \leqq r \leqq a_{n+2} - 1.$$

Then

$$\frac{1}{q_{n,r} + q_{n,r+1}} < |q_{n,r}\alpha - p_{n,r}|.$$

Proof. By the lemma of Theorem 10, we must check if

$$\frac{1}{(2r + 1)q_{n+1} + 2q_n} < \frac{\alpha_{n+2} - r}{\alpha_{n+2}q_{n+1} + q_n}.$$

This inequality is equivalent with

$$1 < \left(1 - \frac{r}{\alpha_{n+1}}\right)(2r + 1)\frac{\left(q_{n+1} + \dfrac{q_n}{r + \frac{1}{2}}\right)}{\left(q_{n+1} + \dfrac{q_n}{\alpha_{n+2}}\right)},$$

and is implied by

$$1 \leqq \left(1 - \frac{r}{\alpha_{n+1}}\right)(2r + 1).$$

The same kind of trick used in the proof of Theorem 10 now shows that this inequality is satisfied for $0 \leqq r \leqq a_{n+2} - 1$, as was to be shown.

A solution p/q with relatively prime integers p and q, $q \geqq 1$, of the inequality $|q\alpha - p| < \omega(q)/q$, with some positive function ω, will be called an ω-**convergent** of α. The 1-convergents of α are therefore the reduced fractions p/q, with $q \geqq 1$, of the fundamental inequality $|q\alpha - p| < 1/q$.

We order the 1-convergents by increasing denominators. By Theorem 10, the sequence of denominators looks like

$$\cdots < q_{n+1} < q_{n,1}(?) < q_{n,a_{n+2}-1}(?) < q_{n+2} < \cdots$$

and the question marks mean that the intermediate terms $q_{n,1}$ or $q_{n,a_{n+2}-1}$ may or may not be present.

As a special case of Theorem 11, we find:

Corollary. *If p/q, and p'/q' are two successive 1-convergents of α, then*

$$\frac{1}{q+q'} < |q\alpha - p|.$$

This section more or less concludes our study of the formalism of continued fractions. For a discussion of more specialized topics, cf. Perron's excellent book [21]. It is a problem to extend the results of this chapter to simultaneous approximations, those being described by investigating

$$\|q_1 X_1 + \cdots + q_m X_m\|,$$

where X_i is a vector in some higher dimensional space, and the norm is that of the torus. A first attempt was made by Perron [22], and was recently pursued by Bernstein [5], [6], [7]. For a description of possible eventual applications to contexts of algebraic geometry, cf. [17], [18].

CHAPTER II

Asymptotic Approximations

II, §1. DISTRIBUTION OF THE CONVERGENTS

We begin by an old result of Dirichlet.

Theorem 1. *Let α be a real number, and N a positive integer. There exists an integer q, $0 < q \leqq N$ such that $\|q\alpha\| < 1/N$.*

Proof. Cut up the interval $[0, 1]$ into N equal segments of length $1/N$, and consider the $N + 1$ numbers

$$0\alpha, 1\alpha, 2\alpha, \ldots, N\alpha$$

modulo $\mathbf{Z}$. Two of them must lie in the same segment (mod $\mathbf{Z}$), say $r\alpha$ and $s\alpha$ with $r < s$. We let $q = s - r$, and obtain

$$\|q\alpha\| < \frac{1}{N} \leqq \frac{1}{q},$$

as desired.

We are interested in a lower bound for the integer q of Theorem 1. Let α be an irrational number. Let g be a positive function, which will always be assumed to be increasing (not necessarily strictly), and $\geqq 1$. We shall say that α is of **type** $\leqq g$ if for all sufficiently large numbers B, there exists a solution in relatively prime integers q, p of the inequalities

$$|q\alpha - p| < 1/q \qquad \text{and} \qquad B/g(B) \leqq q < B.$$

Theorem 2. *Let $\{p_n/q_n\}$ be the sequence of principal convergents to α, and let f be an increasing function $\geqq 1$ such that for all n sufficiently large,*

$$\frac{1}{q_n f(q_n)} \leqq |q_n\alpha - p_n|.$$

Then α is of type $\leqq f$.

Proof. Since $|q_n\alpha - p_n| \lesssim 1/q_{n+1}$, we conclude that

$$q_{n+1} < q_n f(q_n).$$

Given N large, we find n such that $q_n < N \leqq q_{n+1}$. Then

$$\frac{N}{f(N)} \leqq \frac{N}{f(q_n)} < q_n < N,$$

thereby proving that α is of type $\leqq f$.

Theorem 2 admits a partial converse, which shows that a type for α determines some kind of lower bound for $|q\alpha - p|$ with q, p relatively prime.

Theorem 3. *Let α be of type $\leqq g$. Assume that the function $t/g(t)$ is strictly increasing, and let g^* be its inverse function. Then for any sufficiently large integral solution q, p of $|q\alpha - p| < 1/q$, with q, p relatively prime, we have*

$$\frac{1}{q + g^*(q)} < |q\alpha - p|.$$

Proof. Let p/q be a 1-convergent of α, and let p'/q' be the 1-convergent of α with smallest denominator $> q$. Then by hypothesis,

$$\frac{q'}{g(q')} \leqq q < q',$$

whence $q' \leqq g^*(q)$. By the Corollary of Theorem 11, Chapter I, §4, we conclude that

$$\frac{1}{q + g^*(q)} \leqq \frac{1}{q + q'} < |q\alpha - p|,$$

as was to be shown.

Remark 1. Suppose that g is a function which grows reasonably slowly, namely such that there exists a constant $c \geqq 1$ for which

$$g^*(t) \leqq ctg(t)$$

for all t sufficiently large. Then we can rewrite our inequality

$$\frac{1}{q(1 + cg(q))} \leqq |q\alpha - p|.$$

This happens when g is constant, or grows like the log, or a power of the log. When $g(t) = t^\epsilon$, then one gets a slightly different estimate.

Remark 2. In Theorem 3, and also in other applications (viz. Chapter III, §2) it is more useful to deal with a variation of the notion of type. Thus we may say that α is of **cotype** $\leqq g$ if given a sufficiently large number B, there exists a solution in relatively prime integers q, p of the inequalities

$$|q\alpha - p| < 1/q \qquad \text{and} \qquad B < q \leqq Bg(B).$$

As we saw in Remark 1, in most applications a type and cotype can be taken as the same function.

To get some idea of possible types for numbers, we shall now prove a simple theorem of Khintchine. We recall that a set of numbers is said to have **measure** 0 if given $\epsilon > 0$, the set can be covered by a countable number of intervals, such that the sum of the lengths of these intervals is $< \epsilon$.

Theorem 4. *Let ψ be a positive function such that*

$$\sum_{q=1}^{\infty} \psi(q)$$

converges. Then for almost all numbers α (i.e. outside a set of measure 0), *there is only a finite number of solutions to the inequality*

$$\|q\alpha\| < \psi(q).$$

Proof. Given $\epsilon > 0$, select q_0 such that

$$\sum_{q \geqq q_0}^{\infty} \psi(q) < \epsilon.$$

We may restrict our attention to those numbers α lying in the interval $[0, 1]$. Consider those for which the inequality has infinitely many solutions. For each $q \geqq q_0$, consider the intervals of radius $\psi(q)/q$ surrounding the rational numbers

$$\frac{0}{q}, \frac{1}{q}, \ldots, \frac{q-1}{q}.$$

Every one of our α will lie in one of these intervals because for such α we have

$$\left|\alpha - \frac{p}{q}\right| < \frac{\psi(q)}{q}.$$

The measure of the union of these intervals is bounded by the sum

$$\sum_{q \leqq q_0} q \frac{2\psi(q)}{q} < 2\epsilon,$$

as was to be shown.

For example, we can take $\psi(q) = 1/q(\log q)^{1+\epsilon}$ for any $\epsilon > 0$. Thus we can take the function $f(t) = (\log t)^{1+\epsilon}$ for almost all numbers in Theorem 2.

Theorem 5. *Let ψ be a positive function such that*

$$\sum_{q=1}^{\infty} \psi(q)$$

diverges. Then for almost all numbers α, there exist infinitely many solutions to the inequality $\|q\alpha\| < \psi(q)$.

We refer to Khintchine's book for the proof of Theorem 5.

Theorems 4 and 5 will be called **Khintchine's convergence and divergence** theorems respectively.

II, §2. NUMBERS OF CONSTANT TYPE

There is a special kind of numbers which provides useful examples, and is especially easy to work with. They are characterized by the properties of the next theorem.

Theorem 6. *The following properties concerning an irrational number α are equivalent.*

CT 1. *There exists a constant $c > 0$ such that for all integers $q > 0$ we have $\|q\alpha\| > c/q$.*

CT 2. *For any positive function ψ with convergent sum $\sum \psi(q)$, the inequality*

$$\|q\alpha\| < \psi(q)$$

has only a finite number of solutions.

CT 3. *There exists a constant* $c > 0$ *such that, given a sufficiently large integer* N, *there exists a relatively prime solution* q, p *of the inequality* $|q\alpha - p| < 1/q$, *and* $N < q < cN$. *(In other words,* α *is of constant type.)*

CT 4. *If* $[a_0, a_1, a_2, \ldots]$ *is the continued fraction of* α, *then there exists a constant* $c > 0$ *such that* $a_n < c$ *for all* n.

Proof. Assume CT 1, and suppose that ψ is a function such that $\|q\alpha\| < \psi(q)$ has infinitely many solutions. Then $c/q < \psi(q)$ for infinitely many q. We contend that the sum $\sum \psi(q)$ diverges.

Dividing ψ by c, we may assume without loss of generality that $c = 1$. Let $q_1 < q_2 < \cdots$ be the increasing sequence of q such that $\psi(q_n) > 1/q_n$. Define $\varphi(q) = 1/q_n$ for $q_{n-1} < q \leqq q_n$. Then $\varphi \leqq \psi$, and it suffices to prove that $\sum \varphi(q)$ diverges. Then

$$\sum \psi(q) \geqq \sum \varphi(q) \geqq \frac{1}{q_1} + (q_2 - q_1)\frac{1}{q_2} + (q_3 - q_2)\frac{1}{q_3} + \cdots.$$

Take $n = n_1$ large. The first n terms of this series have a lower bound given by

$$(q_2 - q_1)\frac{1}{q_n} + \cdots + (q_n - q_{n-1})\frac{1}{q_n} = \frac{q_n - q_1}{q_n}.$$

Thus for n large, we get a contribution $> \frac{1}{2}$ to our sum. We repeat this procedure with a number n_2 which will give a contribution greater than

$$\frac{q_{n_2} - q_{n_1}}{q_{n_2}} > \frac{1}{2}$$

to our sum, and so on with $n_3, \ldots$. In this manner, we see that the sum diverges, and CT 2 is proved.

Assume CT 2. We shall prove that α satisfies CT 1 by an argument due to Schanuel. Suppose that α does not satisfy CT 1. Then we can find a sequence of integers q_i with

$$1 < q_1 < q_2 < \cdots$$

such that $\|q_i\alpha\| < 1/2^i q_i$. Let

$$\psi(t) = \sum_{i=1}^{\infty} \frac{e}{2^i q_i} e^{-t/q_i}.$$

Then $\psi(q_j) > 1/2^j q_j$ for $j = 1, 2, \ldots$ and the sum for ψ converges. This is a contradiction, which proves that α satisfies CT 1.

We observe that Schanuel's function is very smooth, and behaves as well as possible from the point of view of convexity. Thus if CT 2 is assumed only for such functions, it still follows that α satisfies CT 1.

The equivalence of CT 1 and CT 3 is a special case of Theorems 2 and 3. The equivalence of these with CT 4 follows from the fact that at most two n-th intermediate convergents are also 1-convergents, by Theorem 7 of Chapter I, §3. This proves our theorem.

Numbers of constant type are also said to have **bounded partial quotients**, in view of CT 4.

Example. Let D be a positive integer which is not a square, and let $\alpha = a + b\sqrt{D}$ where a, b are integers. Then α is of constant type. This is trivially seen as follows. Suppose that $|q\alpha - p|$ is small, so that $\alpha - p/q$ is small. Let $\alpha' = a - b\sqrt{D}$ be the conjugate of α. Since p/q approximates α very closely, we conclude that $\alpha' - p/q$ is approximately equal to $\alpha' - \alpha$. But $(q\alpha - p)(q\alpha' - p)$ is a non-zero integer, of absolute value $\geqq 1$. If $|q\alpha - p| \leqq c/q$ for some small $c > 0$, then

$$|q\alpha' - p| \geqq q/c.$$

However, $q\alpha' - p$ is approximately equal to $q(\alpha' - \alpha)$. This shows that c cannot be arbitrarily small.

Example. If α is of constant type, and m/n is a rational number $\neq 0$, then $m\alpha/n$ is also of constant type. The easy proof will be left as an exercise to the reader.

In view of Khintchine's divergence theorem, we see that given an integer $n > 0$, the set of numbers α for which there is only a finite number of solutions of the inequality $\|q\alpha\| < 1/nq$ has measure 0. Call this set S_n. If $m > n$ then $S_n \subset S_m$. Every element of S_n is of constant type, and conversely, every number of constant type lies in some S_n. Since the countable union of sets of measure 0 also has measure 0, it follows that the numbers of constant type form a set of measure 0.

No simple example of numbers of constant type, other than the one given above, is known. The best guess is that there are no other "natural" examples.

II, §3. ASYMPTOTIC APPROXIMATIONS

Throughout this section, we let ψ be a positive function $\leqq 1$, decreasing, such that

$$\sum_{q=1}^{\infty} \psi(q)$$

diverges. We let

$$\Psi(N) = \int_1^N \psi(t)\,dt.$$

For each positive integer N and irrational number α, let $\lambda^+_{\alpha,\psi}(N)$ be the number of solutions in integers q, p of the inequalities

$$0 < q\alpha - p < \psi(q) \qquad \text{and} \qquad 1 \leqq q < N.$$

To simplify the notation, we shall omit the + sign, and also we usually omit the indices α, ψ on λ. It is natural to ask for an asymptotic estimate for λ, but such an estimate was proved only recently for almost all numbers. We shall state this result (without proof). We first recall some terminology.

If F, G are two functions of a real variable, and G is positive, we say that they are **asymptotic** and write $F \sim G$ if

$$\lim_{x\to\infty} F(x)/G(x) = 1.$$

We say that $F = O(G)$ if there exists a constant $C > 0$ such that $|F(x)| \leqq CG(x)$ for all x sufficiently large. We say that $F = o(G)$ if

$$\lim_{x\to\infty} F(x)/G(x) = 0.$$

Theorem 7. *For almost all numbers α, we have*

$$\lambda(N) = \Psi(N) + o\big(\Psi(N)\big).$$

A special case of Theorem 7 was first stated by Leveque [19]. The general theorem was proved by Erdös [8] and Schmidt [24]. In this book, we are principally interested in specific numbers, and we shall omit the proof of Theorem 7, but give a partial result (Corollary 3 of Theorem 8 below) consistent with our point of view. We point out, however, that Schmidt obtains further important generalizations, e.g. higher dimensional ones, and also has a very good error term. This is important, because in dealing with specific numbers, the expression of the error term reflects the special nature of the number under consideration in an essential way. For further work on this, cf. also Gallagher [9].

It is a problem to determine specific numbers, and functions ψ for which λ has a similar asymptotic property. For the statement of the next theorem, we introduce some notation. We write $f \succ g$ and say that f is **much larger than** g if there exists a positive function h tending to infinity such that $f = gh$. We also say that g is **much smaller than** f.

Theorem 8. *Let* α *be an irrational number of type* $\leqq g$. *Write* $\psi(t) = \omega(t)/t$. *Assume that* $\omega \succ g$, *that* ω *is increasing to infinity, and that* $\omega(t)^{1/2} g(t)^{1/2}/t$ *is decreasing for all* t *sufficiently large. Then*

$$\lambda(N) = \Psi(N) + O\left(\int_1^N \frac{\omega(t)^{1/2} g(t)^{1/2}}{t}\, dt\right).$$

Remark. If η is a function tending to 0, then one verifies easily that for $N \to \infty$,

$$\int_1^N \frac{\omega(t)\eta(t)}{t}\, dt = o(\Psi(N)).$$

Consequently, since $\omega \succ g$, we see that the error term given in the theorem implies the asymptotic result $\lambda \sim \Psi$.

If α is a number such that $|q\alpha - p| > 1/qf(q)$ for some increasing function f and q, p relatively prime, then we know by Theorem 2 of §1 that we can take $g = f$, whence the asymptotic result holds whenever $\omega \succ f$. We have two interesting special cases:

Corollary 1. *If* α *is of constant type, then*

$$\lambda(N) = \Psi(N) + O\left(\int_1^N \frac{\omega(t)^{1/2}}{t}\, dt\right)$$

for any function $\omega \succ 1$.

Corollary 2. *Let* $0 < a \leqq 1$ *and let* $\omega(t) = at$. *Then* $\lambda(N)$ *is the number of pairs of integers* q, p *satisfying*

$$0 < q\alpha - p < a \quad \text{and} \quad 1 \leqq q < N.$$

We have

$$\lambda(N) = aN + O\left(\int_1^N \frac{g(t)^{1/2}}{t^{1/2}}\, dt\right).$$

The error term is $o(N)$ *if* $g(t)/t$ *tends to* 0 *as* $t \to \infty$.

When ω is as in Corollary 2, then the problem of estimating λ is known as the **equidistribution problem.** It determines the number of integers q such that $q\alpha$ (mod $\mathbf{Z}$) lies in the interval $[0, a]$, satisfying $1 \leqq q < N$. When $\lambda(N)$ is asymptotic to aN, we interpret this as saying that the numbers $q\alpha$ (mod $\mathbf{Z}$) are equidistributed. Corollary 2 determines the connection between this equidistribution problem and the type of the number α, by means of the error term. This particular case had been

considered long ago, notably by Weyl [29], and in a manner more closely related to the point of view taken here, by Ostrowski [20], and Behnke [4]. Instead of working with the type as we have defined it, however, these last-mentioned authors worked with a less efficient way of determining the approximation behavior of α with respect to p/q, whence followed weaker results and more complicated proofs. The function ω, which shows itself to be quite important in the present estimates was introduced in [15].

Theorem 8 also implies a statement about almost all numbers, since we can apply Theorem 4, §1, to these. If $g(t) = (\log t)^{1+\epsilon}$ then the Khintchine convergence theorem implies that almost all numbers are of type $\leqq g$. Thus:

Corollary 3. *Let ω be a positive function such that $\omega \succ \log^{1+\epsilon}$. Then for almost all numbers α (the exceptions being on a set of measure 0, depending on ω), we have $\lambda_{\alpha,\psi} \sim \Psi$.*

The proof of Theorem 8 will involve first a special case of Corollary 2, as in Lemma 1 below.

It is convenient to abbreviate the **remainder** of a number ξ (mod $\mathbf{Z}$) between 0 and 1 by $R(\xi)$. Thus $R(\xi)$ is the unique number $\xi - p$ (with some integer p) such that $0 \leqq \xi - p < 1$.

Lemma 1. *Let $0 < a \leqq 1$. Let p, q be relatively prime integers, such that $|q\alpha - p| < 1/q$. The number of integers n among q consecutive integers such that $R(n\alpha) < a$ is equal to $qa + O(1)$.*

Proof. Write

$$\alpha = \frac{p}{q} + \frac{\delta}{q^2} \qquad \text{with} \quad |\delta| \leqq 1.$$

Let $n = n_0 + \nu$, with $\nu = 1, \ldots, \ldots, q$. Then

$$n\alpha = (n_0 + \nu)\alpha = n_0\alpha + \nu\alpha = n_0\alpha + \frac{\nu p}{q} + \frac{\nu\delta}{q^2}.$$

The rational numbers $\nu p/q$ $(\nu = 1, \ldots, \ldots, q)$ are equal to

$$\frac{0}{q}, \frac{1}{q}, \ldots, \frac{q-1}{q} \quad (\text{mod } \mathbf{Z}),$$

up to a permutation. The error $\nu\delta/q^2$ is bounded by $1/q$. We can write $n_0\alpha = r/q + \epsilon$ for some integer r, $0 \leqq r < q$ and $|\epsilon| \leqq 1/q$. Hence the

numbers $n\alpha \pmod{\mathbf{Z}}$ are precisely the numbers

$$\frac{r+v}{q} + \epsilon_v \pmod{\mathbf{Z}}$$

with $|\epsilon_v| \leqq 2/q$. Up to a permutation, these numbers are nothing but the numbers

$$\frac{\mu}{q} + \epsilon_\mu \pmod{\mathbf{Z}}$$

with $0 \leqq \mu < q$ and $|\epsilon_\mu| \leqq 2/q$. Now we have

$$R\left(\frac{\mu}{q} + \epsilon_\mu\right) = \frac{\mu}{q} + \epsilon_\mu$$

except possibly when $\mu \geqq q - 2$, or $\mu \leqq 2$ which occurs for at most five values of μ. Thus the number of desired integers n is equal to the number of integers μ with $0 \leqq \mu < q$ such that

$$\frac{\mu}{q} + \epsilon_\mu < a,$$

up to a bounded error term. The number of solutions of this inequality is bounded from above by the number of solutions of

$$\frac{\mu}{q} < a + \frac{2}{q}$$

or equivalently, $\mu < qa + 2$, which differs from qa by a bounded error term. Similarly, we obtain a lower bound which also differs from qa by a bounded error term, and thereby prove our lemma.

Lemma 2. *For all N sufficiently large (depending on ω) and all integers q, p relatively prime, $q > 0$, satisfying the inequalities*

$$0 < |q\alpha - p| < 1/q \qquad \text{and} \qquad 1 \leqq q < \frac{Ng(N)^{1/2}}{\omega(N)^{1/2}},$$

we have

$$\lambda(N) - \lambda(N-q) = \int_{N-q}^{N} \psi(t)\, dt + \theta \frac{q\omega(N)^{1/2} g(N)^{1/2}}{N} + \theta_1,$$

where $|\theta| \leqq 4$, $|\theta_1| \leqq c_1$, and c_1 is an absolute constant.

Proof. We note that $\lambda(N) - \lambda(N-q)$ is the number of integers n satisfying

$$0 < R(n\alpha) < \psi(n) \quad \text{and} \quad N - q \leqq n < N.$$

Let

$$E(t) = \frac{\omega(t)^{1/2} g(t)^{1/2}}{t}.$$

We contend that for N sufficiently large,

$$0 \leqq \psi(N-q) - \psi(N) \leqq 2E(N).$$

To see this, note that

$$\begin{aligned} 0 \leqq \psi(N-q) - \psi(N) &= \frac{\omega(N-q)}{N-q} - \frac{\omega(N)}{N} \\ &= \frac{N\omega(N-q) - (N-q)\omega(N)}{N(N-q)}. \end{aligned}$$

We replace $\omega(N-q)$ by $\omega(N)$, making the right-hand side bigger. Similarly, we can replace $N - q$ in the denominator by $N/2$, because $g(t)/\omega(t) \to 0$ as $t \to \infty$. Using the assumption that $q < Ng(N)^{1/2}/\omega(N)^{1/2}$, we see that our contention follows at once.

We conclude that

$$\psi(N) \leqq \psi(n) \leqq \psi(N-q) < \psi(N) + 2E(N).$$

We can determine bounds for $\lambda(N) - \lambda(N-q)$ replacing $\psi(n)$ by $\psi(N) \pm E(N)$ in the inequality $0 < R(n\alpha) < \psi(n)$. By Lemma 1, we obtain

$$\lambda(N) - \lambda(N-q) = q\psi(N) \pm 2qE(N) + O(1).$$

On the other hand, since ψ is decreasing,

$$q\psi(N) \leqq \int_{N-q}^{N} \psi(t)\, dt \leqq q\psi(N-q) \leqq q\psi(N) + 2qE(N).$$

Hence

$$\lambda(N) - \lambda(N-q) = \int_{N-q}^{N} \psi(t)\, dt + \theta q E(N) + \theta_1,$$

thereby proving Lemma 2.

We may now give the main part of the proof. For N sufficiently large, let

$$B = \frac{Ng(N)^{1/2}}{\omega(N)^{1/2}} \leqq N,$$

and select q, p such that $0 < |q\alpha - p| < 1/q$ and $B/g(B) \leqq q < B$. Then trivially, since $g(B) \leqq g(N)$,

$$\frac{N}{\omega(N)^{1/2} g(N)^{1/2}} \leqq q < \frac{Ng(N)^{1/2}}{\omega(N)^{1/2}}$$

and

$$0 < |q\alpha - p| < 1/q.$$

Using the fact that $\omega(t)^{1/2} g(t)^{1/2}/t$ is decreasing, and the left inequality for q, we get

$$\int_{N-q}^{N} \frac{\omega(t)^{1/2} g(t)^{1/2}}{t}\, dt \geqq \frac{q\omega(N)^{1/2} g(N)^{1/2}}{N} \geqq 1.$$

By Lemma 2, it follows that

$$\lambda(N) - \lambda(N-q) = \int_{N-q}^{N} \psi(t)\, dt + \theta_{N,q} \int_{N-q}^{N} \frac{\omega(t)^{1/2} g(t)^{1/2}}{t}\, dt$$

with $|\theta_{N,q}| \leqq c_1 + 5$. Repeating our argument with $N - q$ instead of N, and taking the sum inductively, we find that

$$\lambda(N) = \Psi(N) + \theta \int_{1}^{N} \frac{\omega(t)^{1/2} g(t)^{1/2}}{t}\, dt + O(1),$$

with $|\theta| \leqq c_1 + 5$. This proves our theorem.

Aside from possible quantitative generalizations to higher dimensions, one also faces the problem of describing the asymptotic behavior to the single number α when g grows slower than ω. It is then not necessarily true that λ is asymptotic to Ψ (cf. Chapter IV and especially Chapter V). In the next section, we shall describe a method which may sometimes be applied to such cases.

As for the higher dimensional case, one may begin by considering linear combinations

$$\|q_1\alpha_1 + \cdots + q_m\alpha_m\|$$

from the present point of view. **Khintchine's transference principle** gives a weak relation between the size of the above combination, and simultane-

ous approximations to the numbers $\alpha_1, \ldots, \ldots, \alpha_m$ (which one assumes linearly independent over the rationals). (Cf. for instance, Cassell's book *Introduction to Diophantine Approximations.*) It would be interesting to formulate a quantitative asymptotic transference principle, depending on a generalized notion of type for linear combinations of several numbers.

II, §4. RELATION WITH CONTINUED FRACTIONS

We shall describe here in a formal setting the method used by Adams to determine asymptotic approximations to e and other numbers [1], [2]. A special case will be carried out in Chapter V.

Let

$$\alpha = [a_0, a_1, \ldots]$$

be irrational as usual. From the relation

$$q_{n+1} = a_{n+1}q_n + q_{n-1}$$

we find

$$\frac{q_{n+1}}{q_n} = a_{n+1} + \frac{1}{q_n/q_{n-1}} = a_{n+1}\left(1 + \frac{1}{a_{n+1}a_n + \rho_n}\right)$$

with some $\rho_n > 0$. By induction, it follows that

$$a_1 \cdots a_n \leqq q_n \leqq a_1 \cdots a_n \prod_{\nu=2}^{n} \left(1 + \frac{1}{a_\nu a_{\nu-1}}\right).$$

Let $A(n) = a_1 \cdots a_n$ and let $P(n)$ be the product

$$P(n) = \prod_{\nu=2}^{n} \left(1 + \frac{1}{a_\nu a_{\nu-1}}\right)$$

so that

$$A(n) \leqq q_n \leqq A(n)P(n).$$

Suppose that there exist strictly increasing functions A_* and A^* such that

$$A_*(n) \leqq A(n) \qquad \text{and} \qquad A(n)P(n) \leqq A^*(n).$$

Then we obtain

$$(\star) \qquad A_*(n) \leqq q_n \leqq A^*(n).$$

Let g_* and g^* be the inverse functions of A_* and A^*, respectively.

Let $\lambda_0(N)$ be the number of solutions in relatively prime integers q, p of the inequalities

(1) $$0 < q\alpha - p < 1/q \quad \text{and} \quad 0 < q < N,$$

and let $\lambda(N)$ be the number of solutions of these same inequalities without the restriction that q, p be relatively prime. We wish to find an expression for λ_0 and λ in terms of the continued fraction for α. We shall obtain it under the following:

Assumption. *The only relatively prime solutions q, p of the inequality*

$$\left| q\alpha - p \right| < \frac{1}{q}$$

for q sufficiently large are given by the principal convergents to α.

Given N, let n be such that

$$q_n \leqq N < q_{n+1}.$$

Then from ($\star$) we find $n \leqq g_*(N)$ and $g^*(N) < g^*(q_{n+1}) \leqq n + 1$, or more clearly,

$$g^*(N) - 1 \leqq n \leqq g_*(N).$$

A principal convergent p_ν/q_ν will satisfy $q_\nu\alpha - p_\nu > 0$ if and only if ν is even, by Theorem 5 of Chapter I, §2. Hence by definition, $\lambda_0(q_n) = \frac{1}{2}n + O(1)$, and we find the bounds

(2) $$\boxed{\tfrac{1}{2}g^*(N) - O(1) \leqq \lambda_0(N) \leqq \tfrac{1}{2}g_*(N) + O(1).}$$

To find similar bounds for λ, we have but to count non-relatively prime solutions of our inequality for each ν, $1 \leqq \nu \leqq n$, and sum these. From Corollary 3 of Theorem 3, Chapter I, §1 (with α_{n+2} instead of a_{n+2}, of course), we see that a positive integer k satisfies

$$|kq_\nu\alpha - kp_\nu| < \frac{1}{kq_\nu}$$

if and only if

$$k^2 < \frac{1}{\alpha_{\nu+2}} + \frac{q_{\nu+1}}{q_\nu}.$$

If $a_{\nu+1} \geqq 2$, $1 \leqq \nu \leqq n$, then trivially $kq_\nu < q_{n+1}$. Hence the number of

integers k such that the multiples kq_ν, kp_ν yield a solution of (1) with $kq_\nu < q_{n+1}$ is equal to

$$a_{\nu+1}^{1/2} + O(1).$$

Hence we obtain the intermediate estimate

$$\lambda(q_{n+1}) = \sum_{\substack{\nu=1 \\ \nu \text{ even}}}^{n} a_\nu^{1/2} + O(n).$$

We can then determine $\lambda(N)$ easily from the inequalities

$$\lambda(q_n) \leqq \lambda(N) \leqq \lambda(q_{n+1}),$$

using the bounds for n in terms of $g^*(N)$ and $g_*(N)$, namely:

$$\boxed{\sum_{\substack{\nu=1 \\ \nu \text{ even}}}^{g^*(N)-2} a_\nu^{1/2} - O(g_*(N)) \leqq \lambda(N) \leqq \sum_{\substack{\nu=1 \\ \nu \text{ even}}}^{g_*(N)} a_\nu^{1/2} + O(g_*(N)).} \tag{3}$$

All the terms appearing in the bounds (2) and (3) are expressed entirely in terms of the continued fraction for α, as was our goal. In the applications, the two functions A_* and A^* can be chosen such that g_* and g^* are quite close to each other, and in this way give an asymptotic estimate for $\lambda(N)$.

The estimate of (3) is slightly coarse, due to the presence of the terms $O(g_*(N))$. This is sufficient for the applications to continued fractions like those considered in Chapter V. One could get a somewhat more exact expression by being more careful in counting the k's, satisfying the inequality

$$k^2 < \frac{q_{\nu+1}}{q_\nu} + \frac{1}{\alpha_{\nu+2}} < a_{\nu+1} + \frac{1}{a_\nu} + \frac{1}{\alpha_{\nu+2}}.$$

At present, one does not yet have sufficiently many examples of continued fractions associated with classical numbers to be able to give useful and significant axiomatizations for this counting.

CHAPTER III

Estimates of Averaging Sums

III, §1. THE SUM OF THE REMAINDERS

Let again α be an irrational real number. In considering the values $R(n\alpha)$, it is natural to form the sum

$$\sum_{n=1}^{N} R(n\alpha)$$

and to estimate its order of magnitude. Since one expects the values $R(n\alpha)$ to be somewhat evenly distributed around $\frac{1}{2}$, it is then better to investigate immediately the sum

$$S_N = \sum_{n=1}^{N} \left(R(n\alpha) - \tfrac{1}{2}\right),$$

which gives the average discrepancy between $\frac{1}{2}N$ and the sum of the remainders. We shall estimate this discrepancy in terms of a type for α.

Theorem 1. *Let α be of type $\leqq f$, and assume that the function $f(t)/t$ is decreasing. Then*

$$S_N = O\left(\int_1^N \frac{f(t)}{t}\,dt\right).$$

Proof. Let p, q be relatively prime integers such that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2} \qquad \text{and} \qquad \frac{N}{f(N)} \leqq q < N.$$

Then

$$\alpha = \frac{p}{q} + \frac{\delta}{q^2} \qquad \text{with} \quad |\delta| \leqq 1.$$

We estimate the sum

$$\begin{aligned} S_N - S_{N-q} &= \sum_{N-q+1}^{N} \left(R(n\alpha) - \tfrac{1}{2}\right) \\ &= \sum_{\nu=1}^{q} \left(R\left(N\alpha - \nu\frac{p}{q} - \nu\frac{\delta}{q^2}\right) - \frac{1}{2}\right). \end{aligned}$$

The fractions $\nu p/q$ range over all fractions $0/q, 1/q, \ldots, (q-1)/q$ and hence $S_N - S_{N-q} = O(1)$. In particular, there exists an absolute constant c_1 such that

$$S_N - S_{N-q} = \theta \int_{N-q}^{N} \frac{f(t)}{t}\, dt$$

with $|\theta| \leqq c_1$, because the integral on the right satisfies the lower estimate

$$\int_{N-q}^{N} \frac{f(t)}{t}\, dt \geqq q\frac{f(N)}{N} \geqq 1.$$

Our theorem follows by induction, repeating our procedure replacing N by $N - q$.

The preceding proof is that given in [18]. The sum of Theorem 1 had been considered classically, notably by Hardy–Littlewood [11], Ostrowski [20], Behnke [4], and Hecke [13]. The reader will find connections between this sum and other problems in these papers, especially as concerns the function defined by the Dirichlet series

$$\sum_{n=1}^{\infty} \frac{R(n\alpha)}{n^s}.$$

It would be quite interesting to investigate the analytic properties of this function of the complex variable s, and extend the results of Hardy–Littlewood and Hecke in this direction. This has to be done in connection with special numbers, say algebraic numbers, or e, for instance.

We observe that the error term of Theorem 1 is quite good. For example, if α is of constant type, then the error, i.e. S_N, is of the order of magnitude $\log N$, which is quite small compared with the total number of terms in the sum. Similarly, we know that almost all numbers are of type $\leqq (\log t)^{1+\epsilon}$, so that for such numbers, the sum S_N is of the order of magnitude $(\log N)^{2+\epsilon}$, which is again quite small.

III, §2. THE SUM OF THE RECIPROCALS

For subsequent applications, it is convenient to deal with a variation of the notion of type, and also to work only with the principal convergents to α. Let g be an increasing positive function $\geqq 1$, and B_0 a positive integer $\geqq 10$. We shall say that α is of **principal cotype** $\leqq g$ for all numbers $\geqq B_0$ if given a number $B \geqq B_0$, there exists a principal convergent p_i/q_i to α such that $B < q_i \leqq Bg(B)$. In particular, if p/q and p'/q' are two successive principal convergents to α, and $q \geqq B_0$, then $q' \leqq qg(q)$.

Theorem 2. *Let α be of principal cotype $\leqq g$ for all numbers $\geqq B_0$. Then for all integers $N \geqq B_0$ we have*

$$\sum_{n=1}^{N} \frac{1}{R(n\alpha)} \leqq 2N \log N + 20Ng(N) + K_0,$$

where

$$K_0 \leqq \sum_{n=1}^{B_0 g(B_0)} \frac{1}{R(n\alpha)}.$$

The same estimate holds if we replace $R(n\alpha)$ by $1 - R(n\alpha)$.

Proof. The statement for $1 - R(n\alpha)$ follows just as for $R(n\alpha)$. So we do just the first estimate.

We shall need a lemma, estimating the sum of the reciprocals taken over certain consecutive integers.

Lemma. *Let p/q and p'/q' be two successive principal convergents to α. Let q_0 be an integer, $1 \leqq q_0 \leqq q$. Let n_0 be an integer $\geqq 0$, and assume $n_0 + q_0 < q'$. Then*

$$\sum_{n=n_0+1}^{n_0+q_0} \frac{1}{R(n\alpha)} \leqq q \log q + 10q'.$$

Proof. We write

$$\alpha = \frac{p}{q} + \frac{\delta}{q^2}, \qquad |\delta| \leqq 1,$$

and

$$n = n_0 + \nu \qquad \text{with} \quad \nu = 1, \ldots, q_0.$$

Then

$$n\alpha = n_0\alpha + \nu\alpha = n_0\alpha + \nu\frac{p}{q} + \epsilon_\nu$$

with $|\epsilon_\nu| \leqq 1/q$. The numbers $\nu \dfrac{p}{q}$ (mod $\mathbf{Z}$) are precisely the same as the

numbers

$$\frac{\mu}{q} \pmod{\mathbf{Z}}, \qquad 0 \leqq \mu \leqq q-1,$$

because p, q are relatively prime. Thus we can write

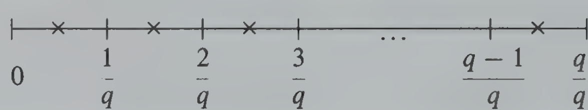
$$n\alpha \equiv n_0\alpha + \frac{\mu}{q} + \epsilon_\mu \pmod{\mathbf{Z}}.$$

(To each n we have associated a unique v and a unique μ, which should be written v_n and μ_n, but we omit the index n for simplicity.) We distinguish cases in estimating our sum.

(a) $n = q$, or n is such that

$$R\left(n_0\alpha + \frac{\mu}{q}\right) \leqq \frac{3}{q} \quad \text{or} \quad \geqq \frac{q-1}{q}.$$

For such n, we see that $R(n_0\alpha + \mu/q)$ occupies the position indicated by a cross in the following diagram:



For such n, we use the fact that $n < q'$, and hence that $1/2q' \leqq R(n\alpha)$, by Corollary 1 of Theorem 6, Chapter I, §2, in other words,

$$\frac{1}{R(n\alpha)} \leqq 2q'.$$

We observe that there are exactly 5 values of n in the present case, and the sum of the reciprocals for these n will be $\leqq 10q'$.

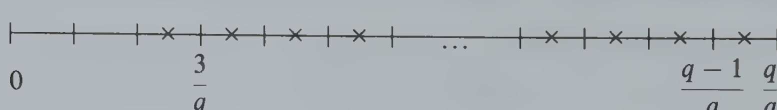
(b) All other n, that is $n \neq q$, and

$$\frac{3}{q} < R\left(n_0\alpha + \frac{\mu}{q}\right) < \frac{q-1}{q}.$$

Then $R(n\alpha) \geqq 2/q$, and in this case the remainders

$$R\left(n_0\alpha + \frac{\mu}{q}\right)$$

lie in all the small intervals indicated by a cross in the following diagram:



Then

$$R\left(n_0\alpha + \frac{\mu}{q}\right) - \epsilon_\mu \leqq R\left(n_0\alpha + \frac{\mu}{q} + \epsilon_\mu\right) = R(n\alpha).$$

We estimate $1/R(n\alpha)$ replacing $R(n\alpha)$ by the fraction m/q lying immediately to the left. If we take the sum of $1/R(n\alpha)$ for all n satisfying our condition (b), we see that this sum is

$$\leqq \sum_{\mu=2}^{q} \frac{1}{\mu/q} \leqq q \log q.$$

Adding the upper bounds in cases (a) and (b) yields an estimate which proves our lemma.

We may now carry out the proper part of the proof of Theorem 2, by induction on $N \geqq B_0$. Let p/q and p'/q' be successive principal convergents to α such that

$$q \leqq N < q'.$$

If $q < B_0$, then $N \leqq B_0 g(B_0)$ and we estimate our sum by K_0 trivially. Suppose $q \geqq B_0$. Then

$$q' \leqq qg(q)$$

by definition of the cotype. We shall distinguish two cases.

Case 1. $N/2 < q$. Then $N < 2q$, and we decompose our sum into two sums:

$$\sum_1^N = \sum_1^q + \sum_{q+1}^N.$$

We apply the lemma to each sum, and find

$$\sum_{n=1}^{N} \frac{1}{R(n\alpha)} \leqq 2q \log q + 20qg(q) \leqq 2N \log N + 20Ng(N),$$

using the monotonicity of log and g.

Case 2. $q \leqq N/2$. From the lemma, we find

$$\sum_{N-q+1}^{N} \frac{1}{R(n\alpha)} \leqq 2q \log q + 20qg(q).$$

(Multiplying by 2 only makes the estimate worse.) By induction,

$$\sum_{1}^{N-q} \frac{1}{R(n\alpha)} \leqq 2(N-q)\log(N-q) + 20(N-q)g(N-q) + K_0.$$

Using the monotonicity of g, we find

$$\begin{aligned}(N-q)g(N-q)+qg(q) &\leqq Ng(N-q)-qg(N-q)+qg(q)\\ &\leqq Ng(N-q) \qquad (\text{because } N-q \geqq q)\\ &\leqq Ng(N),\end{aligned}$$

and similarly with log replacing g. This proves our theorem.

Remark 1. The estimate of Theorem 2 is essentially best possible. Taking $N = q - 1$ or $N = q$ shows that both terms $N \log N$ and $Ng(N)$ are necessary in the estimate. Furthermore, for α of constant type, one sees easily that the sum is $\geqq cN \log N$ for some constant $c > 0$ and all N sufficiently large.

Remark 2. An estimate for K_0 itself can be given only in terms of g. We use the fact that

$$N < q' \leqq Ng(N),$$

and approximate α by p'/q', rather than by p/q. Then we obtain the alternative estimate

$$\sum_{n=1}^{N} \frac{1}{R(n\alpha)} \leqq 10Ng(N)\log\bigl(Ng(N)\bigr)$$

for all $N \geqq B_0$. Thus sums in this chapter can be estimated only in terms of the type—nothing else needs to be known about the number α.

Remark 3. As usual, for almost all numbers we can take

$$g(t) = (\log t)^{1+\epsilon},$$

so that we obtain

$$\sum_{n=1}^{N} \frac{1}{R(n\alpha)} = O\bigl(N(\log N)^{1+\epsilon}\bigr)$$

for almost all numbers. Of course, the constant in O depends on ϵ and α.

Theorem 2 has various applications. We shall first use it to estimate a certain sum involving the sine function. For small values of x, we know that $\sin x$ is approximately equal to x. We need a crude estimate between these two values. To get it, recall that for $0 < y \leqq \pi/2$ we have

$$y - \frac{y^3}{3!} < \sin y < y,$$

whence

$$1 - \frac{y^2}{6} < \frac{\sin y}{y} < 1.$$

The smallest value of the term on the left occurs when $y = \pi/2$, and thus we obtain, in our interval,

$$y < 2 \cdot \sin y.$$

Lemma. *For any number x (real) not equal to an integer, we have*

$$\frac{1}{|\sin \pi x|} < \frac{1}{R(x)} + \frac{1}{1 - R(x)},$$

and hence for irrational α,

$$\sum_{n=1}^{N} \frac{1}{|\sin \pi n\alpha|} < \sum_{n=1}^{N} \frac{1}{R(n\alpha)} + \sum_{n=1}^{N} \frac{1}{1 - R(n\alpha)}.$$

Proof. The expression $|\sin \pi x|$ depends only on $R(x)$, and hence we may assume without loss of generality that $0 < x < 1$. Suppose first that $x = R(x) \leqq \frac{1}{2}$. Let $y = \pi x \leqq \pi/2$. Then by the preceding remark,

$$\frac{1}{|\sin \pi x|} = \frac{1}{|\sin y|} < \frac{2}{y} = \frac{2}{\pi x} < \frac{1}{R(x)}.$$

If, on the other hand, $\frac{1}{2} \leqq R(x) < 1$, then $R(-x) = 1 - R(x)$, and we note that $|\sin \pi x| = |\sin(-\pi x)|$. Thus we can use the same kind of estimate to finish the proof of our lemma.

Theorem 3. *Let the hypotheses be as in Theorem 2, and again $N \geqq B_0$. Then*

$$\sum_{n=1}^{N} \frac{1}{|\sin \pi n\alpha|} \leqq 4N \log N + 40Ng(N) + 2K_0.$$

Proof. This is an immediate consequence of Theorem 2 and the lemma.

III, §3. QUADRATIC EXPONENTIAL SUMS

Let F be a polynomial with real coefficients. It is a classical problem to estimate sums of type

$$\sigma(N, F) = \sum_{n=1}^{N} e^{2\pi i F(n)}.$$

Each term in the sum has absolute value 1, and there are N terms in the sum. However, depending on the special nature of the coefficients of F, one expects a great deal of cancellation to take place in this sum, and one expects the sum to have rather small absolute value compared to N. We shall first prove a result which suggests fruitful generalizations, and is due to Vinogradov (cf. [27], p. 5).

Theorem 4. *Let h be a positive increasing function to infinity. Let k be an integer > 0. Let*

$$\sigma(N, x) = \sum_{n=1}^{N} e^{2\pi i n^k x}.$$

The set of numbers x with $0 \leqq x \leqq 1$ such that

$$|\sigma(N, x)| \geqq N^{1/2}h(N), \qquad \textit{all } N > N_0(x),$$

has measure 0.

Proof. For each positive integer N_0 let A_{N_0} be the set of numbers x such that

$$|\sigma(N, x)| \geqq N^{1/2}h(N), \qquad \text{all } N > N_0.$$

We have

$$|\sigma(N, x)|^2 = \sigma(N, x)\overline{\sigma(N, x)},$$

whence

$$\int_0^1 |\sigma(N, x)|^2\, dx = \sum_{n=1}^{N} \sum_{m=1}^{N} \int_0^1 e^{2\pi i(n^k - m^k)x}\, dx.$$

But for any integer $\mu \neq 0$ we have

$$\int_0^1 e^{2\pi i \mu x}\, dx = 0,$$

whence for $N > N_0$,

$$\begin{aligned} N &= \int_0^1 |\sigma(N, x)|^2\, dx \\ &\geqq Nh(N)^2 \cdot \text{measure}(A_{N_0}). \end{aligned}$$

If measure$(A_{N_0}) \neq 0$, we obtain a contradiction by taking N sufficiently large. Finally, we let $A = \bigcup A_{N_0}$ be the union of all A_{N_0} for

$$N_0 = 1, 2, 3, \ldots.$$

Then A has a measure 0, and A contains the set of numbers x of the theorem.

From Vinogradov's theorem, we see that given $\epsilon > 0$, the absolute value of the sum $\sigma(N, x)$ is $O(N^{1/2+\epsilon})$ for almost all x. This leaves open the problem of giving a similar estimate for specific numbers x. According to our general pattern, such an estimate should arise from a type for x. However, only a special case is known, namely when $k = 2$. To handle this case, we shall reduce the sum to the sum discussed in the preceding section.

Lemma. *Let α, β be real and α irrational. Let*

$$F(n) = \alpha n^2 + \beta,$$

and

$$\sigma(N, F) = \sum_{n=1}^{N} e^{2\pi i F(n)}.$$

Then

$$|\sigma(N, F)|^2 \leqq N + 4 \sum_{n=1}^{N} \frac{1}{|\sin 4\pi n\alpha|}.$$

Proof. We have

$$|\sigma(N, F)|^2 = \sigma(N, F)\overline{\sigma(N, F)} = \sum_{n=1}^{N} \sum_{m=1}^{N} e^{2\pi i(F(n)-F(m))}.$$

In this sum, we consider separately all terms with $m = n$. Each such term is equal to 1, and there are N such terms. The other terms are those with $n < m$ and $m < n$. Thus:

$$\begin{aligned} |\sigma(N, F)|^2 &= N + \sum_{m<n} e^{2\pi i(F(n)-F(m))} + \sum_{m<n} e^{-2\pi i(F(n)-F(m))} \\ &= N + 2 \cdot \mathrm{Re} \sum_{m<n} e^{2\pi i(F(n)-F(m))}, \end{aligned} \tag{1}$$

where Re means real part.

We write $n = m + v$. The map

$$(n, m) \mapsto (n - m, m) = (v, m)$$

gives a bijection of the set of all pairs (n, m) with

$$1 \leqq m < n \leqq N,$$

and the set of all pairs (v, m) with

$$1 \leqq v < N \qquad \text{and} \qquad 1 \leqq m \leqq N - v.$$

Furthermore,

$$F(n) - F(m) = F(m + v) - F(m) = v^2\alpha + 2mv\alpha.$$

Hence

$$\sum_{m<n} e^{2\pi i(F(n)-F(m))} = \sum_{v=1}^{N-1} \sum_{m=1}^{N-v} e^{2\pi i(v^2\alpha)} e^{2\pi i 2mv\alpha}. \tag{2}$$

We shall estimate this last double sum from above by

$$\sum_{v=1}^{N-1} \left| \sum_{m=1}^{N-v} r^m \right|, \qquad \text{where} \quad r = e^{4\pi i v\alpha}. \tag{3}$$

(We note that the term $e^{2\pi i v^2\alpha}$ has absolute value 1, and thus disappears in the estimate.) Now

$$r + r^2 + \cdots + r^M = r(1 + r + \cdots + r^{M-1}) = r\frac{1 - r^M}{1 - r},$$

and for any real x, we have $|\sin x| \leqq |1 - e^{ix}|$. Hence

$$\left| \sum_{m=1}^{N-v} r^m \right| \leqq \frac{2}{|1 - r|} \leqq \frac{2}{|\sin 4\pi v\alpha|}. \tag{4}$$

Using (1), (2), (3), (4) we see that our lemma is proved.

Theorem 5. *Let α, β be real, α irrational, and let*

$$F(n) = \alpha n^2 + \beta.$$

Assume that α is of principal cotype $\leqq g$ for all numbers $\geqq B_0$. Then for $N \geqq B_0$ we have

$$|\sigma(N, F)|^2 \leqq N + 64N \log(4N) + 640Ng(4N) + 4K_0.$$

Proof. We apply the lemma, and Theorem 3, with the obvious inequality

$$\sum_{n=1}^{N} \frac{1}{|\sin 4\pi n\alpha|} \leqq \sum_{n=1}^{4N} \frac{1}{|\sin \pi n\alpha|}.$$

Remark 1. In practice, the cotype g is a slowly increasing function, so that the estimate in Theorem 5 implies in particular

$$\sigma(N, F) = O(N^{1/2+\epsilon})$$

whenever $g(t) = O(t^{\epsilon})$ for some $\epsilon > 0$.

Remark 2. It would be very desirable to generalize Theorem 5 to polynomials of degree > 2. The technique of taking the square is due to Weyl [29]. When it is applied repeatedly to handle polynomials of higher degree, one obtains a rather poor estimate, far removed from the $N^{1/2+\epsilon}$ which Theorem 4 leads one to expect. As for the rest of the proof, Ostrowski was the first to point out the connection between the continued fraction of α and the estimate of the exponential sum [20]. His work was continued by Behnke [4], whose ideas we have used, but combined with new ones to lead to the very sharp estimates of Theorems 2 through 5.

For another approach to the exponential sums, cf. Hardy–Littlewood [11], and for other methods, Vinogradov [27].

For applications to the next section, we derive one more theorem on our exponential sums.

Theorem 6. *Let α, β be real, α irrational. Let m be a positive integer, and*

$$F_m(n) = m\alpha n^2 + \beta.$$

Assume that α is of principal cotype $\leqq g$ for all numbers $\geqq B_0$. Then for $N \geqq B_0$, we have

$$|\sigma(N, F_m)| \leqq C(N \log N)^{1/2}(m \log m)^{1/2} + CN^{1/2}m^{1/2}g(mN)^{1/2} + CK_0^{1/2},$$

where C is an absolute constant.

Proof. We apply the lemma and Theorem 3, just as in the proof of Theorem 5.

Remark. The number β is totally irrelevant for the final estimate, which is uniformly independent of β.

III, §4. SUMS WITH MORE GENERAL FUNCTIONS

Let h be some function, periodic of period 1, and suppose that h is continuous. We can associate with h a Fourier series

$$h(t) \mapsto \sum_{-\infty}^{\infty} c_m e^{2\pi i m t},$$

where c_m is the m-th Fourier coefficient, defined by the scalar product

$$c_m = \int_0^1 h(t)e^{-2\pi imt}\,dt.$$

It is proved in any basic course in analysis that if $c_m = 0$ for all m, then $h = 0$ (this being essentially a variant of the Weierstrass approximation theorem), and thus the Fourier series determines h uniquely. Let us assume in addition that $c_m = O(1/m^2)$ for $m \to \infty$ (the constant in O depending on h). We shall say in that case that the **Fourier coefficients of h tend to 0 like $1/m^2$**. Then it is an exercise to prove that $h(t)$ is equal to the value of its Fourier series for all t (since we assume h continuous). If for instance h is twice continuously differentiable, then its Fourier coefficients satisfy this condition, as one sees at once by integrating by parts twice.

The estimate of Theorem 6 allows us to treat more general sums, as follows. Let $F(n) = \alpha n^2 + \beta$ be as in Theorem 6. We form the sum

$$S(N, h \circ F) = \sum_{n=1}^{N} h \circ F(n) = \sum_{n=1}^{N} h\big(F(n)\big).$$

Using the Fourier series for h, we find that

$$\begin{aligned} S(N, h \circ F) &= \sum_{n=1}^{N} \sum_{-\infty}^{\infty} c_m e^{2\pi imF(n)} \\ &= \sum_{-\infty}^{\infty} c_m \sum_{n=1}^{N} e^{2\pi imF(n)} \qquad (1) \\ &= c_0 N + \sum_{m \neq 0} c_m \sigma(N, mF), \end{aligned}$$

where $\sigma(N, mF)$ is the sum considered in the previous section, and

$$c_0 = \int_0^1 h(t)\,dt.$$

Thus the study of $S(N, h \circ F)$ is reduced to that of the Fourier coefficients c_m, and of our previous sum $\sigma(N, mF)$.

Observe that if α is of principal cotype $\leqq g$, then so is $-\alpha$, because the inequality $|q\alpha - p| < 1/q$ can also be written $|-q\alpha + p| < 1/q$. Thus in estimating $\sigma(N, mF)$ we may take m to be positive. Under the hypotheses of Theorem 6, and the present assumption that the Fourier coefficients of h tend to 0 like $1/m^2$, we find the estimate

$$|S(N, h \circ F) - c_0 N| \leqq C_1 (N \log N)^{1/2} + C_1 N^{1/2} \sum_{m=1}^{\infty} \frac{g(mN)^{1/2}}{m^{3/2}} + C_1 K_0^{1/2}$$

with an absolute constant C_1. When g grows slowly, then the error sum

$$E(N) = \sum_{m=1}^{\infty} \frac{g(mN)^{1/2}}{m^{3/2}}$$

grows correspondingly slowly, and thus the sum $|S(N, h \circ F) - c_0 N|$ differs from $N^{1/2}$ by a slowly growing function. As an example, we state a theorem by selecting special conditions on g.

Theorem 7. *Let h be a continuous function, periodic of period 1, with Fourier coefficients tending to 0 like $1/m^2$. Let c_0 be its 0-th Fourier coefficient. Let $F(n) = \alpha n^2 + \beta$, with irrational α of principal cotype $\leqq g$ for all numbers $\geqq B_0$.*

(i) *If $g(t) = O(t^{\epsilon})$ for every $\epsilon > 0$, $t \to \infty$ then*

$$|S(N, h \circ F) - c_0 N| \leqq KN^{1/2+\epsilon}.$$

(ii) *If $g(t) = O(\log t)^{2k}$ for some number $k > 0$, then*

$$|S(N, h \circ F) - c_0 N| \leqq KN^{1/2}(\log N)^{1/2+k+\epsilon}.$$

(The constant K depends on g, h, and ϵ.)

Proof. This is an immediate consequence of the estimate for $E(N)$ obtained above. The whole point here is that the series

$$\sum_{m=1}^{\infty} \frac{1}{m^s}$$

converges for $s > 1$.

We observe that from a general point of view, the condition that the Fourier coefficients of h tend to 0 like $1/m^2$ is not unnatural: Any sufficiently smooth function will have this property, as we have already remarked.

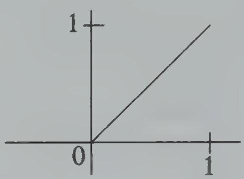
However, in some applications one must deal with functions h which are not continuous, and thus whose Fourier coefficients do not converge to 0 sufficiently rapidly to apply the previous arguments. One important example is the function

$$h(t) = R(t),$$

where R has been considered before: We have $R(t) = t$ for $0 \leqq t < 1$, and extend R by periodicity to all real numbers. In that case, one tries to approximate R by a continuous function whose Fourier coefficients behave better. (Cf. Behnke [4], p. 306.) The Fourier coefficients of R

tend to 0 like $1/m$. If one perturbs R in a δ-neighborhood of 0 to make it continuous, and approximates R by a function R_δ, then the best one sees how to do at the moment is to have the Fourier coefficients of R_δ tend to 0 like $1/m^2\delta$. By this technique, it is then possible to obtain some kind of an estimate for $S(N, R \circ F) - \frac{1}{2}N$, namely with an exponent $\frac{3}{4}$ instead of $\frac{1}{2}$ in situations analogous to those of Theorem 7. We leave it as a problem to the reader to find the right method which will yield the exponent $\frac{1}{2}$. Since the approximation technique is interesting for its own sake, we shall carry out the arguments giving the partial result.

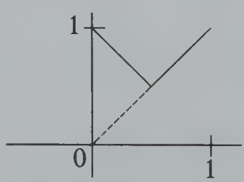
The graph of R between 0 and 1 looks like this:



For each small positive number δ, we shall define two functions U_δ and L_δ such that for all t we have

$$L_\delta(t) \leqq R(t) \leqq U_\delta(t).$$

These functions L_δ and U_δ will be continuous, quite close to R except in a δ-neighborhood of 0, and will have Fourier coefficients tending to 0 like $1/m^2\delta$. We let U_δ be the function whose graph looks like this:



In order words,

$$U_\delta(t) = \begin{cases} 1 + \left(1 - \dfrac{1}{\delta}\right)t, & 0 \leqq t \leqq \delta, \\ t, & \delta \leqq t \leqq 1. \end{cases}$$

For other t, U_δ is defined by periodicity. Then U_δ is continuous. Furthermore, if we denote its Fourier coefficients by $c_m(\delta)$, then by elementary integrations, we find

$$c_0(\delta) = \frac{1}{2} + \frac{\delta}{2}$$

and for $m \neq 0$,

$$c_m(\delta) = \frac{1}{(2\pi i)^2} \frac{e^{-2\pi i m\delta} - 1}{m^2\delta},$$

so that in particular, for $m \neq 0$, we have

$$|c_m(\delta)| \leqq \frac{1}{m^2\delta}.$$

We have

$$S(N, U_\delta \circ F) = c_0(\delta)N + \sum_{m\neq 0} c_m(\delta)\sigma(N, mF).$$

We observe that $c_0(\delta)N$ differs from $\frac{1}{2}N$ by δN. On the other hand, by Theorem 6 of §3, we obtain an estimate for

$$c_m(\delta)\sigma(N, mF).$$

We select $\delta = N^{-1/4}$. We then find that

$$|S(N, U_\delta \circ F) - \tfrac{1}{2}N| \leqq N^{3/4}E_1(N)$$

with an error $E_1(N)$ similar to $E(N)$.

One gets a similar estimate with a suitable function L_δ replacing U_δ, and since

$$S(N, L_\delta \circ F) \leqq S(N, R \circ F) \leqq S(N, U_\delta \circ F),$$

we see that the estimates for the two extreme sums imply a similar estimate for $S(N, R \circ F)$.

CHAPTER IV

Quadratic Irrationalities

IV, §1. QUADRATIC NUMBERS AND PERIODICITY

An **algebraic number** α is a number which is a root of a polynomial equation:

$$a_n\alpha^n + a_{n-1}\alpha^{n-1} + \cdots + a_0 = 0,$$

with rational coefficients a_i, not all 0. Multiplying this equation by a common denominator for all a_i, we see that α is also a root of a polynomial equation with integer coefficients, not all 0. In this chapter, we are concerned with a special case of algebraic numbers, namely those which satisfy a quadratic equation (i.e. those which are a root of a polynomial of degree 2).

Let d be a positive integer. The set of numbers

$$x + y\sqrt{d}$$

with rational x, y is a field, as is immediately verified. We shall assume that it is not equal to $\mathbf{Q}$, and thus that d is not a square. Then without loss of generality, *we shall always assume that d is square free, i.e. that the prime factorization of d does not contain the square of any prime number.* We note that $1, \sqrt{d}$ are linearly independent over $\mathbf{Q}$ (otherwise $\sqrt{d}$ would be rational), and thus every element of our field has a unique expression of the form $x + y\sqrt{d}$ with $x, y \in \mathbf{Q}$. We denote our field by $\mathbf{Q}(\sqrt{d})$.

If $\alpha = x + y\sqrt{d}$ with $x, y \in \mathbf{Q}$, then we define

$$\alpha' = x - y\sqrt{d},$$

and call α' the **conjugate** of α. The reader will easily verify by direct computation that for $\alpha, \beta \in \mathbf{Q}(\sqrt{d})$ we have

$$(\alpha + \beta)' = \alpha' + \beta' \quad \text{and} \quad (\alpha\beta)' = \alpha'\beta'.$$

We define the **trace** of α to be

$$\mathrm{Tr}(\alpha) = \alpha + \alpha',$$

and its **norm** to be

$$\mathbf{N}(\alpha) = \alpha\alpha'.$$

Both the trace and norm are rational numbers (obvious). We note that α is a root of the quadratic polynomial

$$(X - \alpha)(X - \alpha') = X^2 - \mathrm{Tr}(\alpha)X + \mathbf{N}(\alpha).$$

The quadratic equation for α can also be written in the form

$$\alpha^2 - 2x\alpha + (x^2 - dy^2) = 0.$$

We can obtain an equation with integral coefficients, by letting

$$x^2 - dy^2 = \frac{c}{a} \quad \text{and} \quad -2x = \frac{b}{a}$$

where a, b, c are integers, relatively prime, and $a > 0$. Then α is a root of the quadratic equation

$$a\alpha^2 + b\alpha + c = 0$$

with integer coefficients a, b, c, which are relatively prime, $a > 0$, and a, b, c are uniquely determined by these conditions.

We define the **discriminant** of α to be

$$D(\alpha) = b^2 - 4ac = 4a^2y^2d.$$

Since α is real, irrational, we have $D(\alpha) > 0$.

We shall say that α is **reduced** if $\alpha > 1$ and $-1 < \alpha' < 0$ (or equivalently, $-1/\alpha' > 1$). If α is reduced, so is $-1/\alpha'$.

Theorem 1. *Given a positive integer D, there is only a finite number of reduced elements of $\mathbf{Q}(\sqrt{d})$ whose discriminant is D.*

Proof. Let α be reduced, and have D as discriminant. We have

$$\alpha = \frac{-b + \epsilon\sqrt{D}}{2a} > 1 \qquad \text{and} \qquad -1 < \frac{-b - \epsilon\sqrt{D}}{2a} < 0,$$

where $\epsilon = 1$ or $\epsilon = -1$. If $\epsilon = -1$, then

$$-b - \sqrt{D} > 0 \qquad \text{and} \qquad -b + \sqrt{D} < 0,$$

a contradiction. Hence $\epsilon = 1$. Then from these same conditions,

$$-b + \sqrt{D} > 2a > b + \sqrt{D}.$$

Hence $b < 0$ and $0 < -b < \sqrt{D}$. Hence $|b|$ is bounded in terms of D, and the preceding inequality shows that a is bounded in terms of D also. Thus finally $|c|$ is bounded, and our theorem is proved.

Lemma 1. *If α has the discriminant $D > 0$, and β is equivalent to α, then β has the same discriminant D. If m is an integer, and $\beta = \alpha + m$, then again β has the same discriminant as α.*

Proof. Direct brute force computations.

Theorem 2. *Let α be real, quadratic, irrational.*

(i) *In the continued fraction*

$$\alpha = [a_0, \ldots, a_{n-1}, \alpha_n],$$

the number α_n has the same discriminant as α for $n \geqq 1$.

(ii) *If α is reduced, then α_n is reduced for all $n \geqq 1$.*

(iii) *If α is not necessarily reduced, then α_n is reduced for all n sufficiently large.*

Proof. From the lemma, we conclude at once that α_n has the same discriminant as α (using the construction of the continued fraction, and Chapter I, §3). This proves (i). Furthermore, $\alpha_n > 1$. If α is reduced, and

$$\alpha = a + \frac{1}{\beta}$$

with an integer a, and $\beta > 1$, then $-1/\beta' = a - \alpha' > 1$, so that β is also reduced. This proves (ii). To prove (iii), note that

$$\alpha'_n = -\frac{q_{n-1}\alpha' - p_{n-1}}{q_n\alpha' - p_n} = -\frac{q_{n-1}}{q_n}\left(\frac{\alpha' - \dfrac{p_{n-1}}{q_{n-1}}}{\alpha' - \dfrac{p_n}{q_n}}\right).$$

For n large, the fractions p_{n-1}/q_{n-1} and p_n/q_n are close to α, and hence $\alpha' - p_{n-1}/q_{n-1}$ and $\alpha' - p_n/q_n$ are close to $\alpha' - \alpha$. Hence the expression on the right is negative. On the other hand,

$$\alpha'_n + 1 = \frac{1}{q_n}\left[q_n - q_{n-1} - \frac{(-1)^n}{q_n\left(\alpha' - \dfrac{p_n}{q_n}\right)}\right].$$

For n large, the expression,

$$\frac{1}{q_n\left(\alpha' - \dfrac{p_n}{q_n}\right)},$$

is small, and hence we see that $\alpha'_n + 1 > 0$. This proves (iii).

Let α be any (real) irrational number. We shall say that its continued fraction

$$[a_0, a_1, \dots]$$

is **periodic** if there exists an integer k such that for all n sufficiently large, $a_{n+k} = a_n$. We shall say that α is **purely periodic** if $a_{n+k} = a_n$ for all n, and we say that k is the **primitive period** if it is the smallest positive integer with this property.

We shall use the following standard notation for periodic continued fractions. We write

$$[a_0, \dots, a_r, \overline{a_{r+1}, \dots, a_{r+k}}]$$

as an abbreviation for

$$[a_0, \dots, a_r, a_{r+1}, \dots, a_{r+k}, a_{r+1}, \dots, a_{r+k}, \dots]$$

with a repetition of $a_{r+1}, \dots, a_{r+k}$, so that for $n \geqq r + 1$ we have $a_{n+k} = a_n$. A continued fraction is purely periodic if and only if, in the preceding notation, it can be written in the form

$$[\overline{a_1, \dots, a_k}].$$

Lemma 2. *Let α be reduced, and let a be an integer. Let*

$$\alpha = a + \frac{1}{\alpha_1}.$$

Then α_1 is reduced if and only if $a < \alpha < a + 1$, that is, a is the greatest integer $\leqq \alpha$.

Proof. If a satisfies the stated condition $a < \alpha < a + 1$, then $\alpha_1 > 1$, and $\alpha_1' = 1/(\alpha' - a)$, so α_1 is reduced since $a \geqq 1$ and $\alpha' < 0$. Conversely, if $\alpha < a$ then $\alpha_1 < 0$ and if $a + 1 < \alpha$ then $\alpha_1 < 1$, so that α_1 cannot be reduced.

We observe that the relationship between α and α_1 in Lemma 2 determines each number from the other. Indeed, we have

$$-\frac{1}{\alpha_1'} = a + \frac{1}{-1/\alpha'}.$$

Thus, we see that a is also the greatest integer $\leqq -1/\alpha_1'$. Furthermore, α', and hence α, is uniquely determined by α_1', and hence also by α_1.

Theorem 3 (Euler–Lagrange). *Let α be real irrational. The continued fraction of α is periodic if and only if α is quadratic. Assume this is the case. Then α is reduced if and only if its continued fraction is purely periodic.*

Proof. Assume that α is quadratic irrational. We know from Theorem 2 that α_n is reduced for large n, and by Theorem 1, combined with Lemma 1, there is only a finite number of possible values for such α_n. Hence for some n and $k > 1$, we have $\alpha_n = \alpha_{n+k}$. From this, the periodicity follows at once. Suppose in addition that α is reduced, and that $\alpha_m = \alpha_{m+k}$ for some $m > 0$, $k \geqq 1$. From Lemma 2, and the remarks following Lemma 2, we conclude that α_{m-1} is uniquely determined by α_m, whence also $\alpha_{m-1} = \alpha_{m+k-1}$. From the definition of the continued fraction, we conclude that it is purely periodic.

Conversely, if the continued fraction is purely periodic, we can write

$$\alpha = [\overline{a_0, \ldots, a_m}] = [a_0, \ldots, a_m, \alpha].$$

From the relation $\alpha = \sigma_m \alpha$ we see at once that α is quadratic, and furthermore, $\alpha = \alpha_n$ for arbitrarily large n. By Theorem 2(iii) it follows that α is reduced. If the continued fraction is merely periodic, with say

$$\alpha = [a_0, \ldots, a_r, \overline{a_{r+1}, \ldots, a_{r+k}}],$$

then

$$\alpha_{r+1} = [\overline{a_{r+1}, \ldots, a_{r+k}}]$$

is purely periodic, whence quadratic. Since α_{r+1} is equivalent to α, it follows that α is quadratic.

EXERCISE

Show that if α is reduced, and

$$\alpha = [\overline{a_0, \ldots, a_{k-1}}],$$

then

$$-\frac{1}{\alpha'} = [\overline{a_{k-1}, \ldots, a_0}].$$

IV, §2. UNITS AND CONTINUED FRACTIONS

Let α be quadratic as before. We shall say that α is an **algebraic integer** if $\mathrm{Tr}(\alpha)$ and $\mathbf{N}(\alpha)$ are integers. If α is an algebraic integer, then its quadratic equation

$$X^2 - \mathrm{Tr}(\alpha)X + \mathbf{N}(\alpha) = 0$$

has integer coefficients, and conversely. We shall write α in the form

$$\alpha = \frac{u + v\sqrt{d}}{2}$$

with $u, v \in \mathbf{Q}$. The reason for this is that we have $\mathrm{Tr}(\alpha) = u$, and

$$\mathbf{N}(\alpha) = \frac{u^2 - dv^2}{4}.$$

Theorem 4. *An element α of $\mathbf{Q}(\sqrt{d})$ is an algebraic integer if and only if u and v are integers, and the following conditions are satisfied:*

$$(\star)\qquad \begin{cases} u \equiv v \pmod 2 & \text{if } d \equiv 1 \pmod 4, \\ u \equiv v \equiv 0 \pmod 2 & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

Proof. If the stated conditions are satisfied, then one verifies at once from the definitions that α is an algebraic integer. Conversely, if α is an algebraic integer, then from its trace we see that u must be an integer. Then

$$-dv^2 = 4\mathbf{N}(\alpha) - u^2$$

is also an integer, and since d is assumed square-free, it follows that v is also an integer. Then

$$u^2 - dv^2 \equiv 0 \pmod 4.$$

From this congruence, we deduce at once that the congruence conditions on u, v stated in the theorem must hold.

Let

$$\begin{cases} \theta = \dfrac{1+\sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4, \\ \theta = \sqrt{d} & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

Then from Theorem 4, we obtain:

Corollary 1. *An element α of $\mathbf{Q}(\sqrt{d})$ is an algebraic integer if and only if α can be written in the form*

$$\alpha = x + y\theta$$

with integers x, y.

Corollary 2. *The algebraic integers in $\mathbf{Q}(\sqrt{d})$ form a ring.*

Proof. From the representation in Corollary 1, it suffices to prove that θ^2 is an algebraic integer, and this is clear.

A **unit** in $\mathbf{Q}(\sqrt{d})$ is an algebraic integer whose inverse is also an algebraic integer. If ω is a unit, then from the relation $\omega\omega^{-1} = 1$, we conclude by taking the norm that $\mathbf{N}(\omega) = \pm 1$. Conversely, if $\omega\omega' = \pm 1$, then ω is a unit, and $\omega^{-1} = \pm\omega'$. Thus the units form the multiplicative group of non-zero algebraic integers whose norm is 1 or -1, i.e. form the set of solutions of the equation (called **Pell's equation**)

$$\frac{u^2 - dv^2}{4} = \pm 1,$$

where u, v are integers, satisfying conditions ($\star$) of Theorem 4.

Let U be the group of units. We map U into $\mathbf{R}^2$ by

$$L\colon \omega \mapsto (\log|\omega|, \log|\omega'|).$$

Then L is a homomorphism of U into the straight line satisfying the equation $X + Y = 0$. The kernel of L consists of those units having absolute value 1, and the quadratic equations for such units have integer coefficients of absolute value $\leqq 2$. Hence there is only a finite number of elements in the kernel of L, which is therefore a finite group, i.e. a group of roots of unity. Since the only roots of unity which are real are ± 1, it follows that the kernel of L is $\{1, -1\}$. Furthermore, in any bounded

region of $\mathbf{R}^2$ there is only a finite number of elements of the image of L, because any bound on $\log|\omega|$ and $\log|\omega'|$ yields a corresponding bound for the integer coefficients of the quadratic equation satisfied by ω. We shall see in a moment that the image of L contains vectors other than the zero vector.

Let ω_1 be a unit $\neq 1, -1$ such that $L(\omega_1)$ is a vector of smallest length in the image of L. Then we contend that every element of the image of L is an integral multiple of $L(\omega_1)$.

Proof. Let $W_1 = L(\omega_1)$, and let W be in the image of L. Then there is a real number t such that $W = tW_1$. Let m be an integer such that $m \leqq t < m + 1$. Then

$$W - mW_1 = (t - m)W_1,$$

and the length of $W - mW_1$ is smaller than the length of W_1. Hence $W - mW_1 = 0$, so $W = mW_1$, as was to be shown.

When we have shown that there exist non-trivial units (i.e. units other than 1 or -1), we therefore obtain the following theorem:

Theorem 5. *Modulo $\{1, -1\}$, the group of units is infinite cyclic. There exists a unique unit ω_1 which generates this group and such that $\omega_1 > 1$.*

Proof. A unit ω_1 such that $L(\omega_1)$ is a generator for the image of L is of course not uniquely determined, and any one of the units

$$\omega_1, -\omega_1, \omega_1^{-1}, -\omega_1^{-1}$$

would also have this property. However, exactly one among these four numbers will be > 1, thus proving our theorem.

We shall now determine non-trivial units. We let D be the discriminant of the number θ defined previously, so that by a direct computation, we find

$$D = \begin{cases} d & \text{if } d \equiv 1 \pmod 4, \\ 4d & \text{if } d \equiv 2, 3 \pmod 4. \end{cases}$$

We shall call D the **discriminant of the field $\mathbf{Q}(\sqrt{d})$**. From Theorem 4, we obtain:

Theorem 6. *An element α of $\mathbf{Q}(\sqrt{d})$ is an algebraic integer if and only if it can be written in the form*

$$\alpha = \frac{u + v\sqrt{D}}{2}$$

with integers u, v such that $u \equiv Dv \pmod 2$. Furthermore, the algebraic integer α is a unit if and only if, in this representation, we have

$$\frac{u^2 - Dv^2}{4} = \pm 1.$$

From the characterization of units in Theorem 6, we shall deduce results of Pell, Euler, and Lagrange.

Theorem 7. *Let α be a reduced number in $\mathbf{Q}(\sqrt{d})$, having D as discriminant [where D is the discriminant of $\mathbf{Q}(\sqrt{d})$]. Let k be a period of its continued fraction, let v be the greatest common divisor of*

$$(q_{k-1}, p_{k-1} - q_{k-2}, p_{k-2})$$

and let

$$u = p_{k-1} + q_{k-2}.$$

Then

$$\omega = \frac{u + v\sqrt{D}}{2}$$

is a unit > 1, and every unit > 1 is of that type. We have

$$\mathbf{N}(\omega) = (-1)^k.$$

Proof. In the notation of Chapter I, §3, we have

$$\alpha = \sigma_{k-1}\alpha$$

since by definition

$$\alpha = [a_0, \ldots, a_{k-1}, \alpha] = \frac{p_{k-1}\alpha + p_{k-2}}{q_{k-1}\alpha + q_{k-2}}.$$

Thus α is a root of the equation

$$q_{k-1}\alpha^2 + (q_{k-2} - p_{k-1})\alpha - p_{k-2} = 0,$$

and, as before, we let

$$a\alpha^2 + b\alpha + c = 0$$

be the equation for α with relatively prime coefficients a, b, c and $a > 0$. Then we can write

$$q_{k-1} = av, \qquad q_{k-2} - p_{k-1} = bv, \qquad -p_{k-2} = cv. \tag{1}$$

As in the theorem, let

$$u = p_{k-1} + q_{k-2}. \tag{2}$$

Then

$$\begin{aligned} p_{k-1} &= \frac{u - bv}{2} \quad \text{and} \quad q_{k-1} = av, \\ p_{k-2} &= -cv \quad \text{and} \quad q_{k-2} = \frac{u + bv}{2}. \end{aligned} \tag{3}$$

From the relation

$$q_{k-1}p_{k-2} - p_{k-1}q_{k-2} = (-1)^{k-1}$$

we conclude that

$$u^2 - Dv^2 = (-1)^k 4,$$

whence the norm of ω is equal to $(-1)^k$. Since the trace of ω is equal to u, we conclude that ω is a unit, $\omega > 1$.

Conversely, let

$$\omega = \frac{u + v\sqrt{D}}{2}$$

be a positive unit > 1, with integers u, v. Then $u, v \geqq 1$ (because among the four numbers

$$\pm\frac{u \pm v\sqrt{D}}{2}$$

exactly one is > 1). Let p/q and p'/q' be defined in a way similar to (3), namely,

$$p = \frac{u - bv}{2}, \qquad q = av,$$

$$p' = -cv, \qquad q' = \frac{u + bv}{2}.$$

Since $u \equiv vD \pmod 2$ by Theorem 6, and $b \equiv D \pmod 2$ because $D^2 = b^2 - 4ac$, it follows that p, p', q, q' are integers. Then

$$q,\ q' - p,\ -p'$$

are proportional to a, b, c and hence α satisfies the equation

$$q\alpha^2 + (q' - p)\alpha + (-p') = 0,$$

or in other words,

$$\alpha = \frac{p\alpha + p'}{q\alpha + q'}.$$

We have trivially, using the definition $D = b^2 - 4ac$,

$$pq' - qp' = \mathbf{N}(\omega) = \pm 1.$$

Using the inequalities of Theorem 1, §1, we find:

$$\begin{aligned} q' &= \frac{u + bv}{2} > \frac{u - v\sqrt{D}}{2} = \omega' = \frac{\mathbf{N}(\omega)}{\omega} \\ &> \begin{cases} 0 & \text{if } \mathbf{N}(\omega) = 1, \\ -1 & \text{if } \mathbf{N}(\omega) = -1. \end{cases} \\ q - q' &= \frac{(2a - b)v - u}{2} > \frac{v\sqrt{D} - u}{2} = -\omega' = \frac{-\mathbf{N}(\omega)}{\omega} \\ &> \begin{cases} -1 & \text{if } \mathbf{N}(\omega) = 1, \\ 0 & \text{if } \mathbf{N}(\omega) = -1. \end{cases} \end{aligned}$$

Case 1. $\mathbf{N}(\omega) = -1$. Then $q > q' \geqq 0$. If $q' = 0$, then $p' = 1$ and

$$\alpha = p + \frac{1}{\alpha},$$

so that $\alpha = \alpha_1$ and we are in the situation of the first part of the proof, with $k = 1$. If $q' > 0$, then we can apply Theorem 7 of Chapter I, §3, to conclude again that $\alpha = \sigma_{k-1}\alpha$, and that we are in the same situation as the first part of the proof.

Case 2. $\mathbf{N}(\omega) = 1$. Then $q' > 0$ and $q \geqq q'$. If $q = q'$ then $q = q' = 1$ because q, q' are relatively prime. Then a trivial computation yields

$$\alpha = p' + \frac{1}{1 + 1/\alpha}$$

so that we are again in the situation of the first part of the proof, namely $\alpha = [p', 1, \alpha]$. If $q > q'$, then we can apply Theorem 7 of Chapter I, §3, once more to achieve the same result.

We have shown that in all cases, ω is constructed from α as in the first part of the proof, which therefore yields all units > 1.

We shall obtain finally a slightly different description of the units constructed in Theorem 7.

Theorem 8. *Let α be reduced, with discriminant D [the discriminant of $\mathbf{Q}(\sqrt{d})$]. Let k be a primitive period. For each integer $m \geqq 1$ there is a unique unit ω_m such that*

$$\sigma_{mk-1}\begin{pmatrix}\alpha\\1\end{pmatrix} = \omega_m\begin{pmatrix}\alpha\\1\end{pmatrix}$$

(where σ_n is the n-th continued transformation of α). We have $\omega_m = \omega_1^m$, and $\omega_1 > 1$ generates the group of positive units.

Proof. From the definitions, we see that ω_m is precisely the unit constructed in the first part of the proof of Theorem 7, and by Chapter I, §3, we have

$$\sigma_{mk-1} = \sigma_{k-1}^m,$$

whence $\omega_m = \omega_1^m$. This proves our theorem.

We see that the positive units > 1 can be interpreted as eigenvalues of certain linear maps, namely the continued transformations of α corresponding to the periods.

To apply Theorem 7 or 8, one must start with a reduced number α. The next result gives a simple way of finding such a number.

Theorem 9. *Let θ be the number defined for Corollary 1 of Theorem 4. Let $[\theta]$ be the greatest integer $\leqq \theta$. Then*

$$\theta^* = \frac{1}{\theta - [\theta]}$$

is reduced.

Proof. The proof will be left as an exercise.

IV, §3. THE BASIC ASYMPTOTIC ESTIMATE

We have seen in Chapter II, §2, that quadratic numbers are of constant type. Thus the asymptotic estimate of Chapter II, §3, applies, except for the special case of the inequality

$$0 < q\alpha - p < \frac{c}{q}$$

with a constant c. We shall take care of this case by using the special properties of quadratic numbers. We follow [15].

Theorem 10. *Let α be a real quadratic irrational number. Let c be a positive number such that the inequality*

$$0 < q\alpha - p < \frac{c}{q}$$

has infinitely many solutions in integers q, p, and $q > 0$. Let $\lambda(N)$ be the number of solutions of this inequality for $1 \leqq q \leqq N$. Then there exist numbers $c_1, c_2 > 0$ such that for all N we have

$$|\lambda(N) - c_1 \log N| \leqq c_2.$$

In other words,

$$\lambda(N) = c_1 \log N + O(1).$$

The proof will need some lemmas.

Let $\alpha \in \mathbf{Q}(\sqrt{d})$ with $d > 0$, square-free as usual. We let as before

$$\theta = \begin{cases} d & \text{if } d \equiv 2, 3 \pmod 4, \\ \dfrac{1 + \sqrt{d}}{2} & \text{if } d \equiv 1 \pmod 4. \end{cases}$$

Then $1, \theta$ form a basis of the algebraic integers of $\mathbf{Q}(\sqrt{d})$ over $\mathbf{Z}$. We let

$$\alpha = a\theta + b$$

with rational a, b and $a \neq 0$. Let e be a positive integer such that ea and eb are both integers. Then $e\alpha$ is an algebraic integer.

We shall use the phrase "sufficiently large (resp. small)" to mean "greater (resp. smaller) than a constant depending only on θ, α and c".

Lemma 1. *There exists an integer $k > 0$ having the following property. An integer q (sufficiently large) is such that*

$$(\star) \qquad 0 < q\alpha - p < \frac{c}{q}$$

for some integer p if and only if there exists an integer p such that $q\alpha - p$ is positive, sufficiently small, and

$$(\star\star) \qquad |\mathbf{N}(q\alpha - p)| \leqq \frac{k}{e^2}.$$

Proof. We distinguish cases, depending on whether $ce^2(\alpha' - \alpha)$ is or is not an integer.

Suppose that $ce^2(\alpha' - \alpha)$ is not an integer. Then we take

$$k = [ce^2(\alpha' - \alpha)].$$

First assume $(\star)$. Then

$$|\mathbf{N}(qe\alpha - ep)| < \frac{ce^2}{q}|q\alpha' - p| = ce^2\left|\alpha' - \frac{p}{q}\right|.$$

If p/q is close to α, then $\alpha' - p/q$ is close to $\alpha' - \alpha$. Since the norm of an algebraic integer is an integer, we conclude that

$$|\mathbf{N}(qe\alpha - ep)| \leqq k,$$

thereby proving $(\star\star)$.

Secondly, assume $(\star\star)$, and also that $q\alpha - p$ is positive, sufficiently small. Then

$$\begin{aligned} 0 < q\alpha - p &\leqq \frac{k}{e^2|q\alpha' - p|} \\ &\leqq \frac{c}{q}\frac{k}{ce^2\left|\alpha' - \dfrac{p}{q}\right|}. \end{aligned}$$

The quotient $k/ce^2|\alpha' - \alpha|$ is a fixed number < 1. For q sufficiently large, $\alpha' - p/q$ is close to $\alpha' - \alpha$, and hence the right-hand side of our inequality is $< c/q$, thereby proving $(\star)$.

Suppose that $ce^2(\alpha' - \alpha)$ is an integer. We distinguish two subcases, depending on whether $\alpha < \alpha'$ or $\alpha' < \alpha$.

If $\alpha < \alpha'$, then for q sufficiently large, we have

$$\frac{p}{q} < \alpha < \alpha'.$$

We take k as before, and the first part of the argument runs as before, because

$$|\mathbf{N}(qe\alpha - ep)| < ce^2\left(\alpha' - \frac{p}{q}\right).$$

For the second part, we now use the fact that

$$\alpha' - \frac{p}{q} > \alpha' - \alpha,$$

so that

$$0 < q\alpha - p < \frac{c}{q}\frac{k}{ce^2(\alpha' - \alpha)} = \frac{c}{q}.$$

If $\alpha' < \alpha$, then for q sufficiently large, we have

$$\alpha' < \frac{p}{q} < \alpha.$$

This time, we take

$$k = ce^2|\alpha' - \alpha| - 1.$$

In the first part of the argument we have

$$\left|\alpha' - \frac{p}{q}\right| < |\alpha' - \alpha|,$$

and the desired conclusion follows. The second part of the argument is carried out as before, thereby proving the lemma.

Remark. Any $c \geqq 1$ in Theorem 10 will do. Furthermore, Lemma 1 and the fact that the norm of an algebraic integer must be an integer show precisely how small we can take c and still get infinitely many solutions.

In view of Lemma 1, we must count the number of integers q satisfying $1 \leqq q \leqq N$, such that there exists p for which $q\alpha - p$ is positive, sufficiently small, and

$$|\mathbf{N}(qe\alpha - ep)| \leqq k.$$

Let m be an integer, $1 \leqq m \leqq k$. We shall prove that our asymptotic estimate holds for the number of solutions of

$$|\mathbf{N}(qe\alpha - ep)| = m, \qquad 1 \leqq q \leqq N,$$

with $q\alpha - p$ positive, sufficiently small, provided that there is at least one solution. Adding up these estimates for $m = 1, \ldots, k$ and using the fact that our original inequality actually has infinitely many solutions, we obviously obtain a proof of our theorem.

We shall see that our problem can be reduced to counting certain units. *For the rest of the proof*, it is convenient to define a certain equivalence relation. Let ξ, η be algebraic integers. We shall say that they are *equivalent* if there exists a positive unit ω such that $\xi = \omega\eta$. This is obviously an equivalence relation.

Lemma 2. *Given a number $B > 0$, there exists only a finite number of inequivalent algebraic integers ξ such that $|\mathbf{N}(\xi)| \leqq B$.*

Proof. Let ω_0 be a generator > 1 for the positive units. If $|\xi\xi'| \leqq B$, we shall see that there is some power ω_0^r such that

$$|\omega_0^r \xi| \qquad \text{and} \qquad |(\omega_0^r \xi)'|$$

are bounded in terms of B. Since there is only a finite number of algebraic integers with bounded conjugates, this will prove what we want. Suppose that one of the two absolute values $|\xi|$, $|\xi'|$ is $> \omega_0$. By symmetry, say $|\xi| > \omega_0$. Let n be the integer such that

$$\omega_0^n \leqq |\xi| < \omega_0^{n+1}.$$

Then

$$|\xi/\omega_0^n| < \omega_0.$$

On the other hand, $|\xi'| \leqq B/|\xi|$, and $1/|\xi| \leqq 1/\omega_0^n$, whence

$$\left|\frac{\xi'}{\omega_0'^n}\right| \leqq \frac{B}{|\omega_0^n \omega_0'^n|} = B.$$

This proves what we wanted (with $r = -n$).

Lemma 3. *Let q_0, p_0 be integers, $q_0 \neq 0$, and let $\xi_0 = q_0 e\alpha - ep_0$. The set of units u such that $u\xi_0$ can be written in the form $qe\alpha - ep$ with integers q, p is a group.*

Proof. Write $a_1 = ea$ and $b_1 = eb$. We write a unit u as $u = x\theta + y$, with integers x, y. We shall prove that the condition stated in the lemma is equivalent with a congruence condition on x.

We have

$$\xi_0 = q_0 a_1 \theta + q_0 b_1 - ep_0.$$

Then

$$u\xi_0 = \big(x(q_0 b_1 - ep_0) + yq_0 a_1\big)\alpha + xq_0 a_1 d + y(q_0 b_1 - ep_0),$$

and we must find a necessary and sufficient condition that this expression is of type

$$qa_1\alpha + qb_1 - ep,$$

with integers q, p. This amounts to the pair of conditions

$$x(q_0 b_1 - ep_0) + yq_0 a_1 = qa_1,$$
$$xq_0 a_1 d + y(q_0 b_1 - ep_0) = qb_1 - ep.$$

The first one simply means that a_1 divides the left-hand side. Let w be the g.c.d. of a_1 and $(q_0 b_1 - ep_0)$. Let $a_1 = wa_0$. Then the first condition is equivalent with $a_0|x$ (provided $q_0 b_1 - ep_0 \neq 0$, a case we leave to the reader). We shall write $x = a_0 x^*$.

The first condition being satisfied, our second condition yields another divisibility condition, namely that e divides

$$x^*\left(a_0 q_0 a_1 d - b_1 \frac{q_0 b_1 - ep_0}{w}\right).$$

Let t be the g.c.d of e and the expression in parentheses which we have just obtained. Write $e = te_0$. Then our last condition amounts to

$$e_0 | x^*.$$

Hence finally, our two conditions are equivalent with the divisibility

$$a_0 e_0 | x.$$

Since $u^{-1} = \pm u'$, it now follows at once that the units satisfying our divisibility condition form a group, as contended.

Lemma 4. *In Lemma 3, assume that $\xi_0 > 0$, and let S be the subgroup of positive units in Lemma 3. Assume that S is infinite cyclic, and let ω be a generator of S, $0 < \omega < 1$. Write a unit $u \in S$ in the form $u = \omega^n$. The subset of S consisting of those units u such that u is sufficiently small, and such that in the expression $u\xi_0 = qe\alpha - ep$ we have $q > 0$, is of one of the following types: It is empty, it consists of all sufficiently large n, or all sufficiently large even n, or all sufficiently large odd n.*

Proof. We have

$$u\xi_0 - u'\xi_0' = qe(\alpha - \alpha').$$

When u is small, then $u\xi_0$ is also small, and u' is large (positive or negative) because $\mathbf{N}(u) = uu' = \pm 1$. Hence $-u'\xi_0'$ is of the same order of magnitude as $qe(\alpha - \alpha')$. Suppose first that $\alpha > \alpha'$. Then $q > 0$ if and only if $-u'\xi_0' > 0$ whenever u is sufficiently small. Let $\epsilon = \omega\omega'$, and let $\eta = -\xi_0'$. The condition $-u'\xi_0' > 0$ is then equivalent with the condition $\epsilon^n \eta > 0$ (write $u' = \omega'^n$ and multiply the inequality by the positive number ω^n). We then have four cases to consider, depending on whether $\epsilon = 1$, $\epsilon = -1$, $\eta > 0$, or $\eta < 0$. These four cases obviously give rise to the four possibilities described in the lemma. We note finally that the argument is entirely similar in the case that $\alpha' > \alpha$, thus proving our lemma.

Lemma 5. *Let $q_0 > 0$ and p_0 be integers, and let*

$$\xi_0 = q_0 e\alpha - ep_0$$

be > 0. Let $\lambda_0(N)$ be the number of pairs of integers p, q with

$$1 \leqq q \leqq N$$

such that, if we let $\xi = qe\alpha - ep$, then:

(1) *ξ is equivalent to ξ_0;*
(2) *$\xi > 0$; and*
(3) *ξ is sufficiently small.*

Then there exists a constant $c_0 \geqq 0$ such that $\lambda_0(N) = c_0 \log N + O(1)$.

Proof. Let S be the subgroup of positive units in Lemma 3. If S has only one element, we can take $c_0 = 0$, and $\lambda_0(N)$ is then bounded, i.e. $O(1)$. Assume that S has more than one element. Then S is infinite cyclic. Let ω be a generator with $0 < \omega < 1$. An integer q such that $\xi = qe\alpha - ep$ satisfies the three conditions of the lemma is then determined by ξ, which can be written in the form $\xi = u\xi_0$, with $u = \omega^n$, and we deal with one of the four possible types described in Lemma 4. If the set is empty, we are again done. Suppose this is not the case. Consider for definiteness the case when we deal with all sufficiently large even integers n. From the equation

$$u\xi_0 - u'\xi_0' = qe(\alpha - \alpha'),$$

we see that there exist two constants $k_1, k_2 > 0$ such that

$$k_1 q \leqq |u'| \leqq k_2 q,$$

or in other words,

$$k_1 q \leqq u^{-1} \leqq k_2 q.$$

Given a number $k_3 > 0$, we note that the number of units $u = \omega^n$ such that n is a positive even integer and

$$1 < u^{-1} = (1/\omega)^n \leqq k_3 N$$

is equal to the number of positive even integers n such that

$$n \leqq \frac{\log(k_3 N)}{\log(1/\omega)} = \frac{1}{\log(1/\omega)} \log N + O(1),$$

and is therefore given by

$$c_0 \log N + O(1),$$

where $c_0 = \frac{1}{2}\log(1/\omega)$. [We see that the constant k_3 is absorbed in the error term $O(1)$.]

For positive numbers B, let us denote by $S(B)$ the set of units $u = \omega^n \in S$ such that n is sufficiently large, and $u^{-1} \leqq B$. Then for suitable constants $k_3, k_4 > 0$ we have

$$\operatorname{card} S(k_3 N) \leqq \lambda_0(N) \leqq \operatorname{card} S(k_4 N).$$

Using our preceding estimate, we see that the theorem is proved.

We note that even though an asymptotic estimate was known for almost all numbers, the estimate of Theorem 10 was the first to give a similar result for specific numbers which could be exhibited explicitly.

It is a fundamental problem in the theory of algebraic numbers to extend the results of this chapter to those numbers of degree > 2. As Liouville noticed, if α is an algebraic number of degree n, then one obtains a trivial generalization of the fact that quadratic numbers are of constant type, namely there exists a constant $c > 0$ such that

$$|q\alpha - p| > \frac{c}{q^{n-1}}.$$

The proof is essentially the same as that given for the example of Chapter II, §2. The best-known type for algebraic numbers is given by the Thue–Siegel–Roth theorem [23], but in spite of the difficulties which have been encountered historically to reach this result, one expects that the type $O(t^\epsilon)$ can be replaced by $O(\log t)^\rho$ for some $\rho > 0$, and thus Roth's theorem still appears as rather far removed from a good description of the situation.

For other possible directions in generalizing diophantine approximations to algebraic numbers, cf. Schmidt [26] (for a Liouville type result), and also Bernstein [5], [6], [7], and Hasse–Bernstein [12].

CHAPTER V

The Exponential Function

V, §1. SOME CONTINUED FUNCTIONS

It is a general problem to determine the continued fractions for values of classical functions suitably normalized. We shall describe a solution of this problem in a very special case which will allow us in particular to get the continued fraction for e.

We start with the function

$$f(c, x) = 1 + \frac{1}{c}x + \frac{1}{c(c+1)}\frac{x^2}{2!} + \frac{1}{c(c+1)(c+2)}\frac{x^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{1}{c(c+1)\cdots(c+n-1)}\frac{x^n}{n!}.$$

The number c is taken for the moment to be any real number not equal to any integer $0, -1, -2, \ldots$ so that the series is defined. Then it is trivial to verify that

$$f(c, x) = f(c+1, x) + \frac{x}{c(c+1)}f(c+2, x),$$

so that

$$\frac{f(c+1, x)}{f(c, x)} = \frac{f(c+1, x)}{f(c+1, x) + \dfrac{x}{c(c+1)}f(c+2, x)}$$

and hence

$$\frac{f(c+1,x)}{f(c,x)} = \frac{1}{1 + \dfrac{x}{c(c+1)} \dfrac{f(c+2,x)}{f(c+1,x)}}.$$

This almost looks like a continued fraction, but the 1 is on the wrong side in the denominator. Thus we transform this expression by letting $x = z^2$, and write

$$\frac{z}{c}\frac{f(c+1,z^2)}{f(c,z^2)} = \frac{1}{\dfrac{c}{z} + \dfrac{z}{c+1} \dfrac{f(c+2,z^2)}{f(c+1,z^2)}}.$$

This is now in the form where we can express it in terms of the formalism of Chapter I, §1, in a convenient way, namely, by induction:

$$\frac{z}{c}\frac{f(c+1,z^2)}{f(c,z^2)} = \left[0, \frac{c}{z}, \frac{c+1}{z}, \ldots, \frac{c+n}{z}, \alpha_{n+2}\right],$$

where

$$\alpha_{n+2} = \frac{c+n+1}{z} \frac{f(c+n+1,z^2)}{f(c+n+2,z^2)}.$$

We shall therefore obtain the continued fraction expansion for the value of the left-hand side if we can substitute special values for c and z such that $(c+n)/z$ is an integer $\geqq 1$, and such that the last term α_{n+2} is $\geqq 1$, for all integers $n \geqq 0$. For instance, it is immediately verified that the values

$$c = \frac{1}{2} \quad \text{and} \quad z = \frac{1}{2y}$$

for any integer $y \geqq 1$ will fulfill these conditions. In this way, we obtain the **Lambert continued fraction** (which actually was already known to Euler). For these values of c and z, the function can be expressed in a more familiar way. Indeed, we have

$$\begin{aligned} e^w - e^{-w} &= 2w\left(1 + \frac{w^2}{3!} + \frac{(w^2)^2}{5!} + \cdots\right) = 2w \sum_{n=0}^{\infty} \frac{(w^2)^n}{(2n+1)!} \\ &= 2wf\left(\frac{3}{2}, \frac{w^2}{4}\right) \end{aligned}$$

because the coefficient of $(w^2)^n$ in both power series is

$$\frac{1}{\frac{3}{2}(\frac{3}{2}+1)\cdots(\frac{3}{2}+n-1)4^n n!} = \frac{1}{3\cdot 5\cdots(2n+1)2\cdot 4\cdots 2n}$$

$$= \frac{1}{(2n+1)!}.$$

Similarly,

$$e^w + e^{-w} = 2f\left(\frac{1}{2}, \frac{w^2}{4}\right).$$

We obtain the Lambert continued fraction if we let $w = 1/y$:

Theorem 1. *For every integer $y \geqq 1$, we have*

$$\frac{e^{1/y} - e^{-1/y}}{e^{1/y} + e^{-1/y}} = [0, y, 3y, 5y, \ldots]$$

and, in particular, for $y = 2$,

$$\frac{e-1}{e+1} = [0, 2, 6, 10, \ldots].$$

The continued fraction for e will be obtained easily in the next section from the continued fraction for $(e-1)/(e+1)$. Observe that these two numbers are not equivalent, however, since the determinant of

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

is equal to 2.

We note that the same recursive method which we used for the exponential function also works for a wider class of functions (hypergeometric functions, as they are called), including the Bessel function. However, just as in Theorem 1, one obtains continued fraction representations only for special values of the variable, and it is a problem to find the right approach to get the answer in general. There is also the difficulty that for functions satisfying, say, a second-order linear differential equation, like the Bessel function J, one gets information on J'/J, but one still does not have much information concerning the continued fraction of J itself (for rational or integral variables).

V, §2. THE CONTINUED FRACTION FOR *e*

We shall give Euler's argument to prove:

Theorem 2. *The continued fraction of e is*

$$e = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \dots].$$

In other words, $a_0 = 2$, *and for* $m \geqq 1$,

$$a_{3m} = a_{3m-2} = 1 \qquad \text{and} \qquad a_{3m-1} = 2m.$$

Proof. Let r_n/s_n be the principal convergents to the number

$$\alpha = \frac{e+1}{e-1}.$$

Inverting the continued fraction of Theorem 1, we know that

$$\alpha = [2, 6, 10, \dots],$$

and obviously,

$$e = \frac{\alpha + 1}{\alpha - 1}.$$

Let $\xi = [2, 1, 2, 1, 1, 4, 1, 1, 6, 1, 1, \dots]$, and let p_n/q_n be the principal convergents to ξ. We shall prove that for all $n \geqq 0$ we have

$$(\star) \qquad p_{3n+1} = r_n + s_n \qquad \text{and} \qquad q_{3n+1} = r_n - s_n.$$

These relations are verified by a direct computation for $n = 0, 1$. By definition, we have for $n \geqq 2$:

$$r_n = (2 + 4n)r_{n-1} + r_{n-2},$$
$$s_n = (2 + 4n)s_{n-1} + s_{n-2}.$$

On the other hand, we multiply the recursion formulas for p_n, q_n by the number indicated on the right in the following table, and add.

$$\begin{array}{ll|r}
p_{3n-3} &= p_{3n-4} + p_{3n-5} & 1, \\
p_{3n-2} &= p_{3n-3} + p_{3n-4} & -1, \\
p_{3n-1} &= 2np_{3n-2} + p_{3n-3} & 2, \\
p_{3n} &= p_{3n-1} + p_{3n-2} & 1, \\
p_{3n+1} &= p_{3n} + p_{3n-1} & 1.
\end{array}$$

We obtain (for $n \geqq 2$) formulas similar to those for r_n, s_n, namely

$$p_{3n+1} = (2 + 4n)p_{3n-2} + p_{3n-5},$$
$$q_{3n+1} = (2 + 4n)q_{3n-2} + q_{3n-5}.$$

Relations ($\star$) now follow by induction. We conclude that

$$\frac{p_{3n+1}}{q_{3n+1}} = \frac{r_n + s_n}{r_n - s_n} = \frac{\dfrac{r_n}{s_n} + 1}{\dfrac{r_n}{s_n} - 1}.$$

We know that p_{3n+1}/q_{3n+1} approaches ξ as n tends to infinity. Since r_n/s_n approaches α, it follows that $\xi = e$, as was to be shown.

By a similar method, one can get the continued fraction for $e^{2/y}$ with an integer $y \geqq 1$. Cf. Perron [21].

V, §3. THE BASIC ASYMPTOTIC ESTIMATE

The asymptotic result of Chapter II, §3, is valid only under certain conditions determined by the type of the number. We shall now reproduce a result of Adams [1], determining the basic asymptotic estimate for e. We let λ be the function $\lambda^+_{e,1/q}$ considered in Chapter II, and we let $\lambda_0(N)$ be the number of solutions in *relatively prime* integers p, q of the inequalities

(1) $$0 < qe - p < 1/q \qquad \text{and} \qquad 1 \leqq q \leqq N.$$

To begin with, we can sharpen Theorem 10 of Chapter I, §4.

Theorem 3. *Any solution p/q in relatively prime integers with $q > 0$ of the inequality $|qe - p| < 1/q$ is a principal convergent.*

Proof. We must show that a solution of the form

$$\frac{p_n + rp_{n+1}}{q_n + rq_{n+1}}$$

with $r = 1$ or $r = a_{n+2} - 1$ cannot occur. If $n = 3m - 2$ or $n = 3m - 1$, then $a_{n+2} = 1$ and so $r = 0$. Suppose $n = 3m$. By the lemma of Theorem 10, Chapter I, §4, we must show that the only solution of

$$\frac{e_{3m+2} - r}{q_{3m} + e_{3m+2}q_{3m+1}} < \frac{1}{q_{3m} + rq_{3m+1}}$$

is $r = 0$. That $r \neq 1$ follows from

$$e_{3m+2} > a_{3m+2} = 2(m+1) \quad \text{and} \quad q_{3m+1}/q_{3m} = [1, 1, 2m, \ldots] < 2.$$

One sees that $r \neq a_{3m+2} - 1$ in a similar way, thereby proving Theorem 3.

In what follows, we shall meet the function $4^x\Gamma(x + \frac{3}{2})$, which is strictly increasing for $x > 0$. We shall denote its inverse function by g. Simple estimates show that $g(x)$ is asymptotic to

$$(\log x)/\log \log x.$$

Theorem 4. *We have*

$$\lambda_0(N) = \tfrac{3}{2}g(N) + O(1)$$

and

$$\lambda(N) = \tfrac{1}{6}(2g(N))^{3/2} + O(g(N)).$$

Proof. We shall first obtain a special case of Theorem 4, for special values of N.

Lemma 1. *We have for all $n, m \geqq 1$,*

$$\lambda_0(q_n) = \tfrac{1}{2}n + O(1),$$

$$\lambda(q_{3m+1}) = \tfrac{1}{6}(2m)^{3/2} + O(m).$$

Proof. The first assertion is a direct consequence of Theorem 3, the definitions, and the fact that the principal convergents alternate on both sides of e by Theorem 5 of Chapter I, §2. (This gives rise to the factor $\frac{1}{2}$ in front of n.) As for the second assertion, we must determine the multiples of the convergents p_n, q_n which are also solutions of (1), and $0 < q_n\alpha - p_n$. Using once more the Lemma of Theorem 10, Chapter I, §4, with $r = 0$, we see that $q = kq_n$ and $p = kp_n$ is a solution of (1) if and only if

$$\frac{e_{n+2}}{q_n + e_{n+2}q_{n+1}} < \frac{1}{k^2 q_n} \quad \text{or} \quad k^2 < \frac{1}{e_{n+2}} + \frac{q_{n+1}}{q_n}.$$

If $n = 3m - 1$ or $n = 3m$, then the condition implies $k^2 < 4$, so $k = 1$ is the only possibility. If $n = 3m - 2$, the condition amounts to

$$k^2 < 2m + O(1), \quad \text{i.e.} \quad 1 \leqq k < (2m)^{1/2} + O(1).$$

For such k, we note that $kq_{3m-2} < q_{3m+1}$ (for m sufficiently large). Hence

modulo $O(m)$, we find

$$\lambda(q_{3m+1}) \equiv \sum_{\substack{\nu=0 \\ \nu \text{ even}}}^{m-1} (2\nu)^{1/2} \equiv \int_0^{m/2} \sqrt{2}(2x)^{1/2}\, dx = \tfrac{1}{6}(2m)^{3/2}.$$

(We restricted the sum to ν even because we consider only those n such that $0 < q_n e - p_n$, and those are the even n, so that here this amounts to even ν.) Our lemma is proved.

Proving Theorem 3 now essentially amounts to obtaining m as a function of q_{3m-2}.

Lemma 2. *There exist constants $c_1, c_2 > 0$ such that*

$$c_1 4^m \Gamma(m + \tfrac{3}{2}) \leqq q_{3m+1} \leqq c_2 4^m \Gamma(m + \tfrac{3}{2}).$$

Proof. We note that the equations

$$\begin{aligned} q_{3m+2} &= 2(m+1)q_{3m+1} + q_{3m}, \\ q_{3m+1} &= q_{3m} + q_{3m-1}, \\ q_{3m} &= q_{3m-1} + q_{3m-2}, \end{aligned}$$

can be solved to yield

$$\frac{q_{3m+1}}{q_{3m-2}} = 2(2m+1) + \frac{q_{3m-5}}{q_{3m-2}},$$

so that

$$\frac{q_{3m+1}}{q_{3m-2}} = [2(2m+1), 2(2m-1), \ldots].$$

Thus

$$q_{3m+1} \geqq 2(2m+1)q_{3m-2} \geqq 2^2(2m+1)(2m-1)q_{3m-5} \geqq \cdots$$

and hence

$$q_{3m+1} \geqq c_1 4^m \Gamma(m + \tfrac{3}{2})$$

is clear. Conversely,

$$\begin{aligned} \frac{q_{3m+1}}{q_{3m-2}} &\leqq 2(2m+1) + \frac{1}{2(2m-1)} \\ &\leqq 2(2m+1)\left(1 + \frac{1}{4(2m+1)(2m-1)}\right) \end{aligned}$$

and proceeding inductively we see that

$$q_{3m+1} \leqq 2^m(2m+1)(2m-1)\cdots \prod_{\nu=1}^{m}\left(1+\frac{1}{4(2\nu+1)(2\nu-1)}\right),$$

so

$$q_{3m+1} \leqq c_2 4^m \Gamma(m+\tfrac{3}{2}),$$

where c_2 is determined by the infinite product.

To prove the theorem, we find to any given N the integer m such that $q_{3m-2} \leqq N < q_{3m+1}$. Thus

$$c_1 4^{m-1}\Gamma(m-1+\tfrac{3}{2}) \leqq N < c_2 4^m \Gamma(m+\tfrac{3}{2}),$$

and consequently

$$g(N/c_2) < m \leqq g(N/c_1) + 1.$$

Since $g(x)$ grows like $(\log x)/\log\log x$, we conclude that

$$m = g(N) + O(1),$$

whence Theorem 4 follows from Lemma 1.

Remark. The procedure here differs only very slightly from that in Chapter II, §4, and we could simply have applied what we did before. We chose to repeat the arguments *ab ovo*, as in [1], partly to show how one obtains a slightly more accurate bound for q_n $(n = 3m+1)$ in Lemma 2 than by the procedure of Chapter II, which would have introduced an extra factor 2^n. We also remark that we could use the function x^x instead of $\Gamma(x)$, since factors c^x (with c constant > 0) do not affect the asymptotic behavior of the inverse function of $c^x x^x$.

From the continued fraction of e we can also determine a type.

Theorem 5. *If $g(x)$ is as above, the inverse function of $4^x\Gamma(x+\frac{3}{2})$, then e is of type $\leqq 2g + O(1)$.*

Proof. Given N we find q_n such that

$$q_{n-1} < N \leqq q_n.$$

We have

$$\frac{q_n}{q_{n-1}} = a_n + \frac{q_{n-2}}{q_{n-1}} \leqq a_n + 1.$$

Hence

$$\frac{N}{a_n+1} < \frac{q_n}{a_n+1} \leqq q_{n-1} < N.$$

If $n = 3m$ or $3m - 2$, then $a_n = 1$. If $n = 3m - 1$, then from the expression $m = g(N) + O(1)$ obtained previously, we find

$$a_n = a_{3m-1} = 2m = 2g(N) + O(1).$$

This proves what we wanted.

We note that the type of Theorem 5 is essentially best possible.

In [2], Adams has extended the estimate of Theorem 4. He treats the case of more general functions $\psi(t) = \omega(t)/t$ as described in Chapter II, in the range where the result of Theorem 8, Chapter II, §3, does not apply. He also treats a wider class of numbers, those whose continued fractions look like that of e (including Hurwitz continued fractions—cf. Perron [21]).

The general problem is to investigate values of the classical functions, suitably normalized, from the present point of view. (Cf. [17] for a general discussion.) The function e^t is of course the simplest one. The first problem which arises, and probably the simplest, is to determine the continued fraction for e^a where a is rational (or even an arbitrary integer). One wants to see how the special analytic properties of one of the classical functions are reflected in the arithmetical properties of their values. Theorem 1 of §1 gives an example of this reflection, showing that all numbers obtained as values of a function suitably normalized, and over a suitable domain of definition, have similar continued fractions.

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APPENDIX A

Some Computations in Diophantine Approximations[1]

By W. ADAMS at Berkeley and S. LANG at New York

Let $w_1, \ldots, w_m$ be (real) numbers, linearly independent over the rationals. Let B be an integer > 0 and c a number > 0. We define $\lambda(B, c)$ to be the number of solutions of the inequality

$$|q_1 w_1 + \cdots + q_m w_m| \leqq \frac{c}{\max |q_i|^{m-1}}$$

with integers q_i ($i = 1, \ldots, m$) such that $|q_i| \leqq B$. The theory of diophantine approximations is concerned among other things with the study of this function $\lambda(B, c)$. For special numbers, nothing seems to be known about it except for quadratic irrationalities [1]. On the other hand, there is a theorem of Schmidt [2] stating that for almost all m-tuples, there is a number c_1 such that $\lambda(B, c)$ is asymptotic to $c_1 \log B$.

We have carried out computations giving this function for large values of B (up to 10^4 or 10^6) and a few classical numbers like e, π, $\log 2$, γ, $e + \pi$, $(\sqrt{5} - 1)/2$. These computations in each case have a tendency to confirm the expected behavior, and we thought it would be useful to have them available in the literature. Most of them are for the usual case $|qw + p|$ (i.e. $m = 2$).

Of course, one knows the asymptotic theorem for $(\sqrt{5} - 1)/2$ [1], but we have included it for comparison with the other numbers, about which nothing is known.

One could ask for the behavior of another function $\tau(B, c)$, equal to

[1] We are much indebted to Columbia University for use of the machine, IBM 794, Project No. UR7HB01.
Reprinted from *J. reine angew. Math.* **220** (1965), pp. 163–173.

the number of solutions of the inequality

$$|q_1 w_1 + \cdots + q_m w_m| \leqq \frac{c}{B}$$

with integers q_i such that $|q_i| \leqq B$. This is a more delicate function, and we merely observe that in all cases which we have computed below, its values are small, depending on c, ranging between 2 and 8, except in the case of π, where there is a well-known disturbance corresponding to one unusually good approximation with $q = -113$ and $p = 355$.

The tables are easy to read. *On the top of each table*, we indicate the value for $w_1, w_2, \ldots$ and c. In case we deal with $q_1 w_1 + q_2$, we write it as $qw + p$.

The columns from left to right give:

The values for $q_1, q_2, \ldots$ (or q, p).

In the third column, the maximum number N such that the absolute value of the sum is $\leqq c/N^{m-1}$.

The absolute value of the sum itself. The symbols E-01 at the end of a number mean that this number should be multiplied by 10^{-1}. Similarly, E-02 means multiplication by 10^{-2}, etc.

Finally, in the last three columns, we give certain values for B, $\lambda(B, c)$, and $\lambda(B, c)/\log B$. Since c is fixed at the top of the table, we write $\lambda(B)$ instead of $\lambda(B, c)$. Values of B were selected, for which a new solution for the inequality occurs.

The quotient $\lambda(B)/\log B$ is the one which should be more or less constant. Its variations seem to be small, and also seem to follow some wave pattern. It is hard to tell whether there is anything significant to this. We have computed it by slide rule, and rounded it off to one decimal.

If one plots the graph of $\lambda(B)$ against B, one finds that it is a step function, which fits a curve $c_1 \log B$ rather well.

For simplicity of programming, a few uninteresting solutions of the inequality in some cases have been omitted from the first four columns. These always occur at the beginning, with values of $q_1, q_2, \ldots$ equal to 0, 1, 2, or 3. However, their existence was taken into account when computing the function $\lambda(B)$.

We observe finally that we usually took a number c which is such that given an integer $n > 0$ there always exists a solution for an inequality (say)

$$|qw + p| < \frac{c}{n}$$

with $|q| \leqq n$. This allows one to make a check on the computations to see that the required solution has been found by the machine.

$w = e, \quad c = 1 + e$

q	p	N	$\lvert qw + p\rvert$	B	$\lambda(B)$	$\dfrac{\lambda(B)}{\log B}$
−1	2	5	7.1828182E-01	10	9	3.9
−1	3	13	2.8171817E-01	20	11	3.7
−2	5	8	4.3656365E-01	30	12	3.5
−2	6	6	5.6343634E-01	40	13	3.5
−3	8	24	1.5484548E-01	50	14	3.6
−4	11	29	1.2687268E-01	70	15	3.5
−7	19	132	2.7972799E-02	90	16	3.6
−11	30	37	9.8899887E-02	110	17	3.6
−14	38	66	5.5945598E-02	200	18	3.4
−18	49	52	7.0927087E-02	300	19	3.3
−25	68	86	4.2954288E-02	390	20	3.4
−32	87	248	1.4981489E-02	580	21	3.3
−39	106	286	1.2991310E-02	1080	22	3.2
−71	193	1868	1.9901794E-03	1270	23	3.2
−110	299	337	1.1001130E-02	1460	24	3.3
−142	386	934	3.9803587E-03	2730	25	3.2
−213	579	622	5.9705381E-03	4180	26	3.1
−394	1071	1222	3.0404128E-03	5450	27	3.1
−465	1264	3540	1.0502334E-03	8170	28	3.1
−536	1457	3955	9.3994594E-04	20510	29	2.9
−1001	2721	33714	1.1028750E-04	23230	30	3.0
−1537	4178	4481	8.2965844E-04	25950	31	3.1
−2002	5442	16857	2.2057501E-04	49180	32	3.0
−3003	8163	11238	3.3086251E-04	75120	33	3.0
−7543	20504	22141	1.6793342E-04	98350	34	3.0
−8544	23225	64502	5.7645921E-05	147520	35	3.0
−9545	25946	70633	5.2641582E-05	468490	36	2.8
−18089	49171	743012	5.0043345E-06	517660	37	2.8
−27634	75117	78054	4.7637251E-05	566830	38	2.9
−36178	98342	371506	1.0008669E-05	1084490	39	2.8
−54267	147513	247670	1.5013007E-05	1651310	40	2.8
−172346	468485	489086	7.6025026E-06	2168970	41	2.8
−190435	517656	1431122	2.5981571E-06			
−208524	566827	1545336	2.4061301E-06			
−398959	1084483	19357453	1.9208528E-07			
−607483	1651310	1679450	2.2139866E-06			
−797918	2168966	9678726	3.8417056E-07			

$w = \pi, \quad c = 1 + \pi$

q	p	N	$\lvert qw + p \rvert$	B	$\lambda(B)$	$\dfrac{\lambda(B)}{\log B}$
−1	3	29	1.4159265E-01	10	10	4.3
−1	4	4	8.5840733E-01	20	11	3.7
−2	6	14	2.8318531E-01	30	13	3.8
−3	9	9	4.2477795E-01	50	14	3.6
−6	19	27	1.5044408E-01	70	15	3.5
−7	22	467	8.8514247E-03	90	16	3.6
−8	25	31	1.3274123E-01	340	17	3.0
−14	44	233	1.7702850E-02	360	18	3.1
−21	66	155	2.6554275E-02	380	19	3.2
−28	88	116	3.5405699E-02	710	20	3.1
−106	333	469	8.8212804E-03	1070	21	3.1
−113	355	137391	3.0144353E-05	1420	22	3.0
−120	377	466	8.8815690E-03	1780	23	3.1
−226	710	68695	6.0288706E-05	2130	24	3.1
−339	1065	45797	9.0433060E-05	2490	25	3.2
−452	1420	34347	1.2057741E-04	2840	26	3.3
−565	1775	27478	1.5072176E-04	3200	27	3.3
−678	2130	22898	1.8086612E-04	3550	28	3.4
−791	2485	19627	2.1101047E-04	3910	29	3.5
−904	2840	17173	2.4115483E-04	4260	30	3.6
−1017	3195	15265	2.7129917E-04	4620	31	3.7
−1130	3550	13739	3.0144353E-04	4970	32	3.8
−1243	3905	12490	3.3158788E-04	5330	33	3.9
−1356	4260	11449	3.6173224E-04	5680	34	3.9
−1469	4615	10568	3.9187659E-04	6040	35	4.0
−1582	4970	9813	4.2202094E-04	6390	36	4.1
−1695	5325	9159	4.5216529E-04	6750	37	4.2
−1808	5680	8586	4.8230965E-04	104000	38	3.3
−1921	6035	8081	5.1245400E-04	104400	39	3.4
−2034	6390	7632	5.4259835E-04	208400	40	3.3
−2147	6745	7231	5.7274271E-04	312700	41	3.2
−33102	103993	216504	1.9129322E-05	521100	42	3.2
−33215	104348	375994	1.1015029E-05	625400	43	3.2
−66317	208341	510407	8.1142933E-06	833800	44	3.2
−99532	312689	1427780	2.9007206E-06	1146500	45	3.2
−165849	521030	794391	5.2135438E-06	1980200	46	3.2
−199064	625378	713890	5.8014411E-06	2292900	47	3.2
−265381	833719	1790708	2.3128232E-06	3126600	48	3.2
−364913	1146408	7044754	5.8789737E-07			
−630294	1980127	2400864	1.7250422E-06			
−729826	2292816	3522377	1.1757948E-06			
−995207	3126535	3642097	1.1371449E-06			

$w = \gamma, \quad c = 1 + \gamma$

q	p	N	$\|qw + p\|$	B	$\lambda(B)$	$\dfrac{\lambda(B)}{\log B}$
−1	2	10	1.5443133E-01	10	9	3.9
−2	3	5	2.6835300E-01	20	12	4.0
−2	4	5	3.0886266E-01	30	13	3.8
−3	5	13	1.1392167E-01	50	14	3.6
−4	7	38	4.0509654E-02	60	15	3.7
−7	12	21	7.3412020E-02	80	16	3.7
−8	14	19	8.1019307E-02	100	17	3.7
−11	19	47	3.2902366E-02	130	18	3.7
−15	26	207	7.6072874E-03	150	19	3.8
−26	45	62	2.5295079E-02	250	20	3.6
−30	52	103	1.5214575E-02	280	21	3.7
−41	71	89	1.7687792E-02	400	22	3.7
−56	97	156	1.0080504E-02	520	23	3.7
−71	123	637	2.4732171E-03	790	24	3.6
−86	149	307	5.1340703E-03	1190	25	3.5
−142	246	318	4.9464341E-03	1580	26	3.5
−157	272	592	2.6608532E-03	4870	27	3.2
−228	395	8405	1.8763610E-04	5260	28	3.3
−299	518	690	2.2855810E-03	5660	29	3.3
−456	790	4202	3.7527221E-04	10520	30	3.2
−684	1185	2801	5.6290831E-04	10920	31	3.3
−912	1580	2101	7.5054441E-04	16170	32	3.3
−2807	4863	7117	2.2158384E-04			
−3035	5258	46460	3.3947739E-05			
−3263	5653	10262	1.5368836E-04			
−6070	10516	23230	6.7895478E-05			
−6298	10911	13171	1.1974063E-04			
−9333	16169	18383	8.5792886E-05			

$w = e + \pi, \quad c = 1 + w$

q	p	N	$\|qw + p\|$	B	$\lambda(B)$	$\dfrac{\lambda(B)}{\log B}$
−1	5	7	8.5987448E-01	10	7	3.0
−1	6	48	1.4012552E-01	20	8	2.7
−2	12	24	2.8025103E-01	40	9	2.4
−6	35	43	1.5924689E-01	50	11	2.8
−7	41	358	1.9121374E-02	90	12	2.7
−8	47	56	1.2100414E-01	260	13	2.3
−14	82	179	3.8242748E-02	300	14	2.5
−43	252	270	2.5397272E-02	340	15	2.6
−50	293	1093	6.2758975E-03	630	16	2.5

q	p	N	$\lvert qw + p\rvert$	B	$\lambda(B)$	$\dfrac{\lambda(B)}{\log B}$
−57	334	534	1.2845477E-02	920	17	2.5
−107	627	1044	6.5695792E-03	1840	18	2.4
−157	920	23358	2.9368167E-04	2760	19	2.4
−314	1840	11679	5.8736333E-04	3680	20	2.4
−471	2760	7786	8.8104500E-04	4600	21	2.5
−628	3680	5839	1.1747267E-03	19613	22	2.2
−785	4600	4671	1.4684083E-03	20533	23	2.3
−3347	19613	63176	1.0858254E-04	40146	24	2.3
−3504	20533	37060	1.8509913E-04			
−6851	40146	89652	7.6516590E-05			

$$w = \log\log 2 + \pi, \quad c = 1 + w$$

q	p	N	$\lvert qw + p\rvert$	B	$\lambda(B)$	$\dfrac{\lambda(B)}{\log B}$
−1	2	4	7.7507973E-01	10	9	3.9
−1	3	16	2.2492026E-01	20	11	3.7
−2	5	6	5.5015946E-01	30	12	3.5
−2	6	8	4.4984053E-01	40	13	3.5
−3	8	11	3.2523920E-01	50	14	3.6
−4	11	37	1.0031893E-01	70	15	3.5
−5	14	30	1.2460133E-01	90	16	3.6
−9	25	155	2.4282403E-02	120	17	3.6
−13	36	49	7.6036528E-02	140	18	3.6
−18	50	77	4.8564805E-02	230	19	3.5
−22	61	72	5.1754125E-02	340	20	3.4
−31	86	137	2.7471723E-02	700	21	3.2
−40	111	1183	3.1893203E-03	810	22	3.3
−49	136	178	2.1093082E-02	920	23	3.4
−80	222	591	6.3786406E-03	1720	24	3.2
−120	333	394	9.5679609E-03	2630	25	3.2
−249	691	733	5.1464809E-03	4350	26	3.1
−289	802	1928	1.9571606E-03	6980	27	3.1
−329	913	3063	1.2321597E-03	11320	28	3.0
−618	1715	5206	7.2500097E-04	15660	29	3.0
−947	2628	7443	5.0715869E-04	22630	30	3.0
−1565	4343	17329	2.1784228E-04	26980	31	3.1
−2512	6971	13048	2.8931642E-04			
−4077	11314	52817	7.1474140E-05			
−5642	15657	25791	1.4636814E-04			
−8154	22628	26408	1.4294828E-04			
−9719	26971	50405	7.4893998E-05			

$w = \log\log 31, \quad c = 1 + w$

q	p	N	$\lvert qw + p\rvert$	B	$\lambda(B)$	$\dfrac{\lambda(B)}{\log B}$
−1	2	2	7.6627796E-01	10	12	5.2
−2	2	4	4.6744407E-01	20	14	4.7
−2	3	4	5.3255592E-01	30	15	4.4
−3	3	3	7.0116611E-01	40	16	4.3
−3	4	7	2.9883389E-01	60	17	4.2
−4	5	34	6.5111854E-02	80	18	4.1
−5	6	13	1.6861018E-01	100	19	4.1
−8	10	17	1.3022371E-01	140	20	4.1
−9	11	21	1.0349833E-01	190	21	4.0
−13	16	58	3.8386472E-02	230	22	4.0
−17	21	83	2.6725382E-02	330	23	4.0
−30	37	191	1.1661090E-02	420	24	4.0
−47	58	148	1.5064292E-02	650	25	3.9
−60	74	95	2.3322180E-02	740	26	3.9
−77	95	656	3.4032016E-03	1070	27	3.9
−107	132	270	8.2578886E-03	1480	28	3.8
−154	190	328	6.8064032E-03	1800	29	3.9
−184	227	460	4.8546870E-03	2540	30	3.8
−261	322	1538	1.4514855E-03	3280	31	3.8
−338	417	1144	1.9517161E-03	4340	32	3.8
−522	644	769	2.9029709E-03	5880	33	3.9
−599	739	4465	5.0023061E-04	7620	34	3.8
−860	1061	2348	9.5125487E-04	10160	35	3.8
−1198	1478	2232	1.0004612E-03			
−1459	1800	4952	4.5102425E-04			
−2058	2539	45394	4.9206364E-05			
−2657	3278	4065	5.4943697E-04			
−3517	4339	5559	4.0181788E-04			
−4116	5078	22697	9.8412728E-05			
−6174	7617	15131	1.4761909E-04			
−8232	10156	11348	1.9682546E-04			

$w = \dfrac{\sqrt{5} - 1}{2}, \quad c = 1, \quad 0 < qw + p$

q	p	N	$\lvert qw + p\rvert$	B	$\lambda(B)$	$\dfrac{\lambda(B)}{\log B}$
−1	0	1	6.1803398E-01	10	3	1.3
−2	1	4	2.3606797E-01	20	4	1.3
−5	3	11	9.0169942E-02	40	5	1.3
−13	8	29	3.4441853E-02	90	6	1.3
−34	21	76	1.3155617E-02	240	7	1.3

q	p	N	$\|qw + p\|$	B	$\lambda(B)$	$\frac{\lambda(B)}{\log B}$
−89	55	199	5.0249987E-03	610	8	1.2
−233	144	521	1.9193787E-03	1600	9	1.2
−610	377	1364	7.3313743E-04	4200	10	1.2
−1597	987	3571	2.8003358E-04			
−4181	2584	9349	1.0696331E-04			

$$w_1 = \log 3, \quad w_2 = \log 2, \quad c = \log 3 + \log 2$$

q	p	N	$\|qw_1 + pw_2\|$	B	$\lambda(B)$	$\frac{\lambda(B)}{\log B}$
−1	2	6	2.8768207E-01	1	6	8.6
−2	3	15	1.1778303E-01	5	8	5.0
−3	5	10	1.6989903E-01	10	10	4.4
−4	6	7	2.3556607E-01	20	13	4.4
−5	8	34	5.2116001E-02	30	14	4.1
−7	11	27	6.5667034E-02	40	15	4.1
−10	16	17	1.0423200E-01	50	16	4.1
−12	19	132	1.3551033E-02	70	17	4.0
−17	27	46	3.8564967E-02	90	18	4.1
−24	38	66	2.7102067E-02	110	19	4.0
−29	46	71	2.5013934E-02	150	20	4.0
−41	65	156	1.1462901E-02	170	21	4.1
−53	84	858	2.0881324E-03	240	22	4.0
−65	103	114	1.5639166E-02	260	23	4.1
−94	149	191	9.3747685E-03	320	24	4.2
−106	168	429	4.1762647E-03	410	25	4.2
−147	233	245	7.2866362E-03	490	26	4.2
−159	252	286	6.2643971E-03	570	27	4.3
−200	317	344	5.1985038E-03	1060	28	4.0
−253	401	576	3.1103715E-03	1540	29	4.0
−306	485	1752	1.0222391E-03	2110	30	3.9
−359	569	1680	1.0658932E-03	3170	31	3.9
−665	1054	41044	4.3654110E-05	4220	32	3.8
−971	1539	1830	9.7858500E-04	5270	33	3.9
−1330	2108	20522	8.7308221E-05	6330	34	3.9
−1995	3162	13681	1.3096233E-04	23680	35	3.5
−2660	4216	10261	1.7461644E-04	24730	36	3.6
−3325	5270	8208	2.1827055E-04	25790	37	3.7
−3990	6324	6840	2.6192466E-04	50510	38	3.5
−14936	23673	28970	6.1848688E-05	75240	39	3.5
−15601	24727	98477	1.8194580E-05	101020	40	3.5
−16266	25781	70376	2.5459532E-05	125750	41	3.5
−31867	50508	246630	7.2649564E-06	176260	42	3.5
−47468	75235	163936	1.0929623E-05	302000	43	3.5

q	p	N	$\lvert qw + pw_2\rvert$	B	$\lambda(B)$	$\frac{\lambda(B)}{\log B}$
−63734	101016	123315	1.4529913E-05	427740	44	3.4
−79335	125743	488928	3.6646670E-06	478250	45	3.5
−111202	176251	497672	3.6002748E-06	603990	46	3.5
−190537	301994	27819424	6.4406776E-08	905990	47	3.4
−269872	427737	480483	3.7290738E-06	1207976	48	3.4
−301739	478245	506736	3.5358826E-06	1509970	49	3.4
−381074	603988	13909712	1.2881355E-07			
−571611	905982	9277333	1.9313302E-07			
−762148	1207976	6954856	2.5762711E-07			
−952685	1509970	5566400	3.2188836E-07			

$$w_1 = \log 3, \quad w_2 = \log 5, \quad w_3 = \log 7, \quad c = \frac{w_1 + w_2 + w_3}{3}$$

q_1	q_2	q_3	N	$\lvert\text{sum}\rvert$	B	$\lambda(B)$	$\frac{\lambda(B)}{\log B}$
−2	0	1	2	2.5131443E-01	10	13	5.7
−2	−1	2	4	8.5157807E-02	20	16	5.3
−1	−3	3	4	8.9195578E-02	30	18	5.3
−8	−3	7	19	4.1589966E-03	40	20	5.4
−3	−4	5	19	4.0377704E-03	50	23	5.9
−11	−7	12	113	1.2122627E-04	60	25	6.1
−6	−8	10	13	8.0755408E-03	70	26	6.1
−757	−8	434	1171	1.1312253E-06	90	27	6.0
−1	−9	8	9	1.6272308E-02	150	28	5.6
−19	−10	19	19	4.2802230E-03	180	29	5.6
−14	−11	17	19	3.9165441E-03	220	30	5.6
−22	−14	24	79	2.4245255E-04	290	31	5.5
−33	−21	36	65	3.6367882E-04	360	32	5.4
−44	−28	48	56	4.8490510E-04	430	33	5.4
−10	−50	47	80	2.4150278E-04	700	34	5.2
−21	−57	59	113	1.2027651E-04	730	35	5.3
−32	−64	71	1278	9.4976534E-07	760	36	5.4
−43	−71	83	112	1.2217604E-04	790	37	5.5
−789	−72	505	2923	1.8146011E-07	830	38	5.7
−64	−128	142	903	1.8995307E-06	860	39	5.8
−821	−136	576	1420	7.6830519E-07			
−96	−192	213	737	2.8492960E-06			
−853	−200	647	950	1.7180705E-06			
−128	−256	284	639	3.7990613E-06			
−160	−320	355	571	4.7488267E-06			
−192	−384	426	521	5.6985919E-06			
3	−2	0	4	7.6961040E-02			
34	−22	−1	46	7.2640789E-04			

q_1	q_2	q_3	N	|sum|	B	$\lambda(B)$	$\frac{\lambda(B)}{\log B}$
5	−1	−2	13	8.1967671E-03			
693	−120	−292	715	3.0307561E-06			
725	−56	−363	863	2.0809908E-06			
−1	2	−1	2	1.7435338E-01			
−1	3	−2	3	1.6211885E-01			
−1	43	−35	65	3.6272906E-04			
−2	5	−3	11	1.2234538E-02			
−7	6	−1	8	2.0431305E-02			
−12	36	−23	56	4.8395534E-04			
−23	29	−11	50	6.0518160E-04			
−174	149	−25	189	4.3001952E-05			

$w_1 = \log 3, \quad w_2 = \log 5, \quad w_3 = \log 7, \quad c = w_1 + w_2 + w_3$

q_1	q_2	q_3	N	|sum|	B	$\lambda(B)$	$\frac{\lambda(B)}{\log B}$
−2	0	1	4	2.5131443E-01	10	32	13.9
−7	0	4	7	9.3354574E-02	20	41	13.7
−2	−1	2	7	8.5157807E-02	30	46	13.5
−4	−1	3	5	1.6615662E-01	40	53	14.4
−0	−2	2	2	6.7294447E-01	50	57	14.6
−1	−2	2	3	4.2566781E-01	60	62	15.1
−2	−2	3	3	4.2163004E-01	70	63	14.8
−4	−2	4	5	1.7031562E-01	80	65	14.8
−6	−2	5	7	8.0998811E-02	90	66	14.7
−13	−2	9	19	1.2355764E-02	100	67	14.6
−1	−3	3	7	8.9195578E-02	110	69	14.7
−8	−3	7	33	4.1589966E-03	120	70	14.6
−1	−4	4	4	2.4727665E-01	130	71	14.6
−3	−4	5	33	4.0377704E-03	150	72	14.4
−5	−5	7	7	8.1120037E-02	160	73	14.4
−0	−6	5	7	7.2923270E-02	170	74	14.4
−16	−6	14	23	8.3179933E-03	180	75	14.4
−11	−7	12	195	1.2122627E-04	190	76	14.5
−6	−8	10	24	8.0755408E-03	210	78	14.6
−757	−8	434	2028	1.1312253E-06	220	81	15.0
−1	−9	8	16	1.6272308E-02	240	82	15.0
−19	−10	19	32	4.2802230E-03	270	83	15.0
−14	−11	17	34	3.9165441E-03	280	84	14.9
−9	−12	15	19	1.2113311E-02	290	85	15.0
−4	−13	13	15	2.0310078E-02	310	86	15.0
−22	−14	24	138	2.4245255E-04	340	87	14.9
−17	−15	22	24	7.9543145E-03	360	90	15.3
−30	−17	31	32	4.4014493E-03	390	91	15.3
−25	−18	29	35	3.7953179E-03	450	92	15.2

q_1	q_2	q_3	N	|sum|	B	$\lambda(B)$	$\frac{\lambda(B)}{\log B}$
−33	−21	36	113	3.6367882E-04	510	93	14.9
−44	−28	48	97	4.8490510E-04	520	94	15.0
−55	−35	60	87	6.0613137E-04	540	95	15.1
−66	−42	72	79	7.2735764E-04	570	97	15.3
−270	−43	188	340	4.0152656E-05	600	98	15.3
−10	−50	47	138	2.4150278E-04	630	99	15.4
−21	−57	59	196	1.2027651E-04	640	101	15.6
−32	−64	71	2213	9.4976534E-07	670	102	15.7
−43	−71	83	195	1.2217604E-04	700	103	15.7
−789	−72	505	4236	1.8146011E-07	730	104	15.8
−54	−78	95	138	2.4340231E-04	760	105	15.8
−65	−85	107	112	3.6462859E-04	790	106	15.9
−31	−107	106	113	3.6177929E-04	830	107	15.9
−302	−107	259	344	3.9202891E-05	860	108	16.0
−42	−114	118	139	2.4055302E-04	890	109	16.0
−53	−121	130	197	1.1932674E-04	920	110	16.2
−64	−128	142	1565	1.8995307E-06	950	111	16.2
−75	−135	154	194	1.2312580E-04			
−821	−136	576	2461	7.6830519E-07			
−334	−171	330	348	3.8253126E-05			
−96	−192	213	1278	2.8492960E-06			
−853	−200	647	1645	1.7180705E-06			
−128	−256	284	1106	3.7990613E-06			
−885	−264	718	1320	2.6678358E-06			
−160	−320	355	989	4.7488267E-06			
−917	−328	789	1134	3.6176010E-06			
−192	−384	426	903	5.6985919E-06			
−949	−392	860	1009	4.5673665E-06			
−224	−448	497	836	6.6483572E-06			
−256	−512	568	782	7.5981227E-06			
−288	−576	639	737	8.5478879E-06			
2	−1	0	2	5.8778666E-01			
2	−2	0	2	1.0216512E-00			
3	−1	−1	4	2.5951119E-01			
12	−7	−1	12	2.8628072E-02			
34	−22	−1	80	7.2640789E-04			
5	−1	−2	23	8.1967671E-03			
8	−3	−2	8	6.8764273E-02			
10	−2	−4	16	1.6393534E-02			
37	−18	−6	37	3.3113625E-03			
565	−376	−8	825	6.8298175E-06			
45	−15	13	74	8.4763416E-04			
56	−8	−25	69	9.6886044E-04			
206	−85	−46	332	4.2052187E-05			
217	−78	−58	242	7.9174088E-05			
597	−312	−79	889	5.8800520E-06			

q_1	q_2	q_3	N	|sum|	B	$\lambda(B)$	$\frac{\lambda(B)}{\log B}$
238	−21	−117	336	4.1102422E-05			
629	−248	−150	971	4.9302868E-06			
661	−184	−221	1081	3.9805214E-06			
693	−120	−292	1239	3.0307561E-06			
725	−56	−363	1495	2.0809908E-06			
−1	2	−1	5	1.7435338E-01			
−1	3	−2	5	1.6211885E-01			
−1	43	−35	113	3.6272906E-04			
−2	2	−1	2	9.2425889E-01			
−2	5	−3	19	1.2234538E-02			
−3	3	−1	3	4.1343327E-01			
−4	4	−1	6	9.7392346E-02			
−4	10	−6	13	2.4469075E-02			
−5	7	−3	8	6.4726502E-02			
−7	6	−1	15	2.0431305E-02			
−9	11	−4	11	3.2665842E-02			
−12	36	−23	98	4.8395534E-04			
−15	32	−18	36	3.5538151E-03			
−23	29	−11	87	6.0518160E-04			
−26	25	−6	36	3.4325888E-03			
−31	26	−4	31	4.7641783E-03			
−35	65	−34	65	1.0891369E-03			
−46	58	−22	62	1.2103632E-03			
−57	51	−10	59	1.3315895E-03			
−110	277	−167	321	4.4901483E-05			
−142	213	−96	325	4.3951718E-05			
−153	206	−84	245	7.7274557E-05			
−163	156	−37	168	1.6422823E-04			
−174	149	−25	328	4.3001952E-05			
−185	142	−13	243	7.8224322E-05			
−327	355	−109	368	3.4272604E-05			
−359	291	−38	363	3.5222370E-05			
−437	632	−276	661	1.0628878E-05			
−469	568	−205	693	9.6791134E-06			
−501	504	−134	730	8.7293481E-06			
−533	440	−63	773	7.7795827E-06			

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APPENDIX B

Continued Fractions for Some Algebraic Numbers

By S. LANG and H. TROTTER at Princeton

The following tables at the end of the paper contain the continued fractions and some related information for a few algebraic numbers, viz:

$$\sqrt[3]{2}, \sqrt[3]{3}, \sqrt[3]{4}, \sqrt[3]{5}, \sqrt[3]{7}, \alpha_1, \alpha_2, \alpha_3,$$

where

$$\alpha_1 = 2\cos\frac{2\pi}{7} \text{ is a root of } x^3 + x^2 - 2x - 1,$$

$$\alpha_2 \text{ is a root of } x^5 - x - 1,$$
$$\alpha_3 = \sqrt[3]{2} + \sqrt{3} \text{ is a root of } x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23.$$

The first numbers were chosen as cubic irrationalities, α_1 as being totally real, and we picked α_2, α_3 at random to be non-cubic. The original motivation was to see if computations for algebraic numbers would be in line with some conjectures made in [3] and [4], and to observe whatever else might come up.

APP. B, 1. TABLE I

In each case, this table is to be read horizontally, and gives the first thousand terms in the continued fraction. For instance, for $\sqrt[3]{2}$, the continued fraction begins

$$[1, 3, 1, 5, 1, 1, 4, 1, 1, 8, \ldots].$$

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The last five terms among these first thousand are $[\ldots, 2, 1, 1, 1, 1]$. Thus the continued fraction $[a_1, a_2, \ldots]$ is laid out in rows of twenty integers, and there are fifty such rows. The position of any a_n is therefore easily determined.

For clarity, and reasons of space distribution, the table is filled only with the 2-digit integers which arose. Whenever a larger integer occurred, it is indicated by the symbols $*a, *b, *c, \ldots,$ and these are then listed separately at the bottom. For instance, for $\sqrt[3]{2}$, we have

$$*a = 534, \qquad *b = 121, \qquad \ldots, \qquad *k = 4941, \qquad \ldots.$$

APP. B, 2. TABLE II

This table gives the frequency count for the digits appearing in the continued fraction. According to a theorem of Kuzmin [2], for almost all numbers α, the probability that the n-th integer a_n in the continued fraction for α is equal to a positive integer k is given by

$$\log_2 \frac{(k+1)^2}{k(k+2)}.$$

For $k = 1$, and almost all numbers, this means that the probability for $a_n = 1$ is approximately 0.41. In each case of the computations, we find that the numbers behave very closely to this generic expectation, thus confirming that in most respects, the continued fractions for algebraic numbers of degree > 2 should be essentially like those of almost all numbers. For instance, among the first thousand a_n for $\sqrt[3]{2}$, $\sqrt[18]{4}$, and α_3 we find that 1 occurs, respectively, 422, 412, and 418 times. The biggest divergence from the Kuzmin number is for $\sqrt[3]{5}$, and even then 433 is still in line with the expected asymptotic estimate. The Kuzmin probability that 2 occurs is approximately 0.17, and again in this case we find that 2 occurs 165 times (resp. 159 times, resp. 168 times) for $\sqrt[3]{2}$, $\sqrt[8]{3}$, α_1, respectively. One should also note that this type of regularity is exhibited throughout, among the first thousand terms.

APP. B, 3. THE OCCURRENCE OF LARGE INTEGERS

Aside from what appear to be routine low numbers, there occur larger numbers which seem to be of two kinds: Some which are only somewhat larger than the ordinary ones, and some which appear to be exceptionally large. For instance, for $\sqrt[3]{2}$, we meet

$$a_{572} = 7451 \qquad \text{and} \qquad a_{620} = 4941.$$

For $\sqrt[3]{5}$, we meet

$$a_{19} = 3052, \qquad a_{691} = 13977, \qquad a_{813} = 49968.$$

The occurrence of large numbers in the continued fractions of certain cubic irrationalities has already been observed by Brillhart, see Churchhouse and Muir [1]. A theoretical explanation for some of them has been proposed by Stark [6], but the following problems remain: Determine whether there is a basic theoretical distinction between what seem to be only medium large numbers, and very large ones. More importantly, determine whether exceptionally large integers will continue to occur throughout the continued fraction, or whether they will stop from occurring. The explanation given by Stark depends on some class numbers being equal to 1, and thus would account for only a finite number of them. In general, the appearance of such large integers may depend on the arithmetic properties of the field obtained from the square root of the discriminant, e.g. its class number. The tables seem to indicate that they stop.

To discuss the statistical significance of exceptionally large values of a_n occurring near the beginning of the sequence of partial quotients, we need an estimate of the probability $q_{N,K}$ that the first N partial quotients of a "random" number are all less than a given integer K. It is perhaps most natural to consider a random number as distributed uniformly on (0, 1), but in this context the distribution given by

$$Pr\{X \leqq c\} = \log_2(1 + c) \qquad \text{for} \quad c \in (0, 1),$$

is more appropriate, because if X has this distribution, then the distribution of the partial quotient $a_n(X)$ is independent of n. In fact,

$$Pr\{a_n(X) < K\} = \gamma_K,$$

where

$$\gamma_K = Pr\{X^{-1} < K\} = 1 - \log_2\left(1 + \frac{1}{K}\right),$$

which is the Kuzmin theorem already alluded to.

To see that this is so, observe that $a_n(X) = [X_n^{-1}]$, where $X_1 = X$, and for $n > 0$,

$$X_{n+1} = X_n^{-1} - [X_n^{-1}].$$

As usual, $[x]$ is the largest integer $\leqq x$. It is then an exercise in calculus to show that if f_n is a density function for X_n, so that

$$Pr\{X_n \leqq c\} = \int_0^c f_n(x)\,dx,$$

then $f_{n+1} = Tf_n$ is a density function for X_{n+1}, where T is the linear operator on $L^1(0, 1)$ defined by

$$(Tf)(x) = \sum_{k=1}^{\infty} (x+k)^{-2} f((x+k)^{-1}).$$

It follows that $f_{n+1} = T^n f_1$. It is easy to verify that the function

$$\frac{1}{(\log 2)(1+x)}$$

is a density function and is invariant under T, so that if X has this density, in which case, by integration,

$$Pr\{X \leqq c\} = \log_2(1+c),$$

all the X_n have the same distribution. In fact, Kuzmin's theorem states that if f is any smooth probability density, then

$$\lim_{n\to\infty} T^n f(x) = \frac{1}{(\log 2)(1+x)}.$$

It follows that as $n \to \infty$ the distribution of $a_n(X)$ tends to the one given above if X has any smooth distribution. For a discussion of all these ideas and a proof of Kuzmin's theorem, see [2].

If the random variables $a_n(X)$ were independent, then we would have

$$q_{N,K} = \gamma_K^N.$$

This is not strictly correct, but can be expected to give a good approximation for large values of N and K. A combination of theoretical and numerical analysis indicated strongly that the relative error is bounded by $\lambda N K^{-2}$, with $\lambda < 1$, and we are confident that the approximation is entirely adequate for our purposes.

The short Table A at the end of this section shows, for each of the numbers investigated, the maximum value A of the first 1000 partial quotients, and the value M for which $a_M = A$. The third column gives

$$p_{1000,A} = 1 - q_{1000,A},$$

the probability that a "random" number would have a value as large as A among its first 1000 partial quotients. The smaller the value of p, the stronger the evidence that the number is unusual. The fourth column gives

$$p_{M,A} = 1 - q_{M,A},$$

the probability of getting a value as large as A among the first M quotients. The value is of course smaller than that of p, and its statistical meaning less clear since M is taken a posteriori to make the probability small.

The table shows that if one goes by the maximum quotient found, only $\sqrt[3]{5}$ appears highly unusual, although one might question $\sqrt[3]{4}$ and α_2. If one also takes into account the second largest quotient, then $\sqrt[3]{5}$ with $a_{691} = 13977$ appears even more unusual, and $\sqrt[3]{2}$ with $a_{620} = 4941$ perhaps comes to be of interest also.

Table A

Number	A	M	$p = 1 - q_{1000,A}$	$p_M = 1 - q_{M,A}$
$\sqrt[3]{2}$	7451	572	0.18	0.10
$\sqrt[3]{3}$	3502	916	0.34	0.31
$\sqrt[3]{4}$	14902	579	0.09	0.05
$\sqrt[3]{5}$	49968	813	0.03	0.02
$\sqrt[3]{7}$	689	611	0.88	0.72
α_1	904	830	0.80	0.73
α_2	11644	588	0.12	0.07
α_3	1446	54	0.63	0.05

APP. B, 4. TABLE III

In each case, this table begins with the columns labeled n, a_n, and q_n. The n indicates n-th position in the continued fraction. The a_n means the n-th partial quotient. The q_n means the denominator in the approximating fraction p_n/q_n (classical notation). For instance, in the case of $\sqrt[3]{2}$, we have

$$a_{36} = 534, \qquad q_{36} = 3.06 \times 10^{19},$$

$$a_{42} = 121, \qquad q_{42} = 8.95 \times 10^{22},$$

and so forth. In machine language, E 19 means multiplication by 10^{19}, and E 486 means multiplication by 10^{486} (the last line in Table III).

Table III includes these data for all n among the first thousand such that $a_n \geqq 50$. We picked 50 as a cutting point after looking at preliminary computations, because it included all the numbers a_n which could be labeled as somewhat large, and at the same time provided only a rather small table.

The last column r_n in Table III gives (up to three decimals) the quotient

$$\frac{q_n}{q_{n-1} \log q_{n-1}}$$

for those values of n when $a_n \geqq 50$. The reason for this quotient to be interesting are as follows. According to a theorem of Roth, if α is algebraic, there is only a finite number of integers $q > 0$, p such that

$$|q\alpha - p| < \frac{1}{q^{1+s}}.$$

It was suggested in [3] and [4] that this theorem should be improvable by an inequality

$$|q\alpha - p| < \frac{1}{qf(q)},$$

where f is a function close to the logarithm, for instance $(\log q)^{1+\epsilon}$, or perhaps even $\log q$ itself, up to a constant factor of course. Such a function is called a type in [4]. If some a_n is small, then q_n/q_{n-1} being approximately equal to a_n shows that the quotient

$$\frac{q_n}{q_{n-1} \log q_{n-1}}$$

is approximately like $1/\log q_{n-1}$. Thus to investigate the possibility of a type f, we look at those n for which a_n is comparatively large. Again n such that $a_n \geqq 50$ seemed to give the most information for the least amount of space used. It is even unsolved for any algebraic number of degree > 2 whether it is of bounded type, but the tables seem to fall fairly well in line with expectations, e.g. differing from the log by a function with a lower order of magnitude (above or below).

We have also programmed the same data for the first 3000 terms of the continued fractions of the cubic numbers listed. In every case, exceptionally large integers did not seem to recur, and generally speaking, the ratio r_n seems to decrease. We thought it pointless to reproduce these more extensive tables in full, but we give in Table B the portion of Table III for $n > 1000$ when $r_n > 1$, rounding off r_n to one decimal.

The tables therefore suggest that the type may in fact not be bigger than a constant times the logarithm, and may even be of an order of magnitude smaller than the logarithm. Following certain asymptotic estimates of Adams, who looked at the continued fraction of e, it was shown (cf. [4]) that the type of e is asymptotic to $\log q/\log\log q$. Thus one is beginning to be accustomed to such small types. Note that for a function essentially not bigger than the log, the series

$$\sum \frac{1}{qf(q)}$$

diverges, so that these cases go very slightly against the Khintchine

convergence principle: If ψ is such that $\sum \psi(q)$ diverges (resp. converges), then for almost all numbers, the inequality

$$|q\alpha - p| < \psi(q)$$

has infinitely many solutions (resp. only a finite number). However, this statistical result is delicate to use for specific numbers with a type in the range of the log, because one also knows that if α is a number such that for every function ψ (decreasing) having convergent sum, the above inequality has only a finite number of solutions, then α must be of bounded type. For all this, cf. [4].

Table B

	n	a_n	q_n	r_n
$\sqrt[3]{2}$	1191	12737	7.74 E 1010	5.5
	2248	2897	2.97 E 1136	1.1
$\sqrt[3]{3}$	1988	2967	3.47 E 1024	1.3
	2407	9559	1.25 E 1242	3.3
$\sqrt[3]{4}$	1974	6368	4.88 E 1010	2.7
	2248	4157	6.92 E 1146	1.6
$\sqrt[3]{5}$	1196	18905	1.47 E 600	13.8
$\sqrt[3]{7}$	None			
$2\cos\frac{2\pi}{7}$	1102	1374	6.84 E 576	1.0

APP. B, 5. COMPUTATIONAL METHOD

The computations were done by the following algorithm, which uses integer arithmetic only, and thus involves no rounding error.

Given a polynomial $p_n(x)$, of degree d, with positive leading coefficient and a unique positive root y_n which is simple, irrational, and greater than 1, we construct a polynomial $P_{n+1}(x)$ as follows. Let $a_n = [y_n]$ be the greatest integer such that $P_n(a_n) < 0$. Define

$$Q_n(x) = P_n(x + a_n) \qquad \text{and} \qquad P_{n+1}(x) = -x^d Q_n(x^{-1}).$$

Then $Q_n(x)$ has exactly one root between 0 and 1, and since the roots of $P_{n+1}(x)$ are the reciprocals of the roots of $Q_n(x)$, we see that $P_{n+1}(x)$ has a unique positive root y_{n+1}. Obviously y_{n+1} is also a simple root, irrational, and greater than 1. Note that the constant term of $Q_n(x)$ is $P_n(a_n) < 0$, so that $P_{n+1}(x)$ also has a positive leading coefficient. Thus $P_{n+1}(x)$ has the properties assumed for $P_n(x)$, and starting from any $P_1(x)$

with these properties, we can define an infinite sequence $P_1(x), P_2(x), \ldots$ with associated positive roots $y_1, y_2, \ldots$.

We have

$$a_n = [y_n] \qquad \text{and} \qquad y_{n+1} = (y_n - a_n)^{-1}.$$

This is precisely equivalent to saying that $a_1, a_2, \ldots$ is the sequence of partial quotients in the continued fraction expansion of y_1. If $P_1(x)$ has integer coefficients, then so has every $P_n(x)$, and the calculation involves only addition and multiplication of integers.

Table I

$\sqrt[3]{2}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	3	1	5	1	1	4	1	1	8	1	14	1	10	2	1	4	12	2	3
20	2	1	3	4	1	1	2	14	3	12	1	15	3	1	4	*a	1	1	5	1
40	1	*b	1	2	2	4	10	3	2	2	41	1	1	1	3	7	2	2	9	4
60	1	3	7	6	1	1	2	2	9	3	1	1	69	4	4	5	12	1	1	5
80	15	1	4	1	1	1	1	1	89	1	22	*c	6	2	3	1	3	2	1	1
100	5	1	3	1	8	9	1	26	1	7	1	18	6	1	*d	3	13	1	1	14
120	2	2	2	1	1	4	3	2	2	1	1	9	1	6	1	38	1	2	25	1
140	4	2	44	1	22	2	12	11	1	1	49	2	6	8	2	3	2	1	3	5
160	1	1	1	3	1	2	1	2	4	1	1	3	2	1	9	4	1	4	1	2
180	1	27	1	1	5	5	1	3	2	1	2	2	3	1	4	2	2	8	4	1
200	6	1	1	1	36	9	13	9	3	6	2	5	1	1	1	2	10	21	1	1
220	1	2	1	2	6	2	1	6	19	1	1	18	1	2	1	1	1	27	1	1
240	10	3	11	38	7	1	1	1	3	1	8	1	5	1	5	4	4	4	7	2
260	1	21	1	1	5	10	3	1	72	6	9	1	3	3	2	1	4	2	1	1
280	1	1	2	1	7	8	1	2	1	8	1	8	3	1	1	3	2	1	8	1
300	1	1	1	1	6	1	4	3	4	1	1	1	4	30	39	2	1	3	8	1
320	1	2	1	3	1	9	1	4	1	2	2	1	6	2	1	1	3	1	4	1
340	2	1	1	5	1	2	10	1	5	4	1	1	4	1	2	1	1	2	12	2
360	1	8	3	2	6	1	3	10	1	2	20	1	6	1	2	*e	2	2	1	2
380	47	1	19	2	2	1	1	1	2	1	1	3	2	8	1	18	3	5	39	1
400	2	1	1	1	1	4	1	5	2	6	3	1	1	1	4	2	1	6	1	1
420	*f	1	3	1	3	1	4	5	1	2	1	13	2	2	2	1	1	1	1	7
440	2	1	7	1	3	1	1	11	1	2	2	4	2	33	3	1	1	2	6	3
460	1	1	3	6	8	3	4	84	1	1	2	1	10	2	2	20	1	3	1	7
480	13	14	1	29	1	1	5	1	7	1	1	2	1	56	1	3	2	1	13	2
500	1	2	2	2	1	1	1	1	1	1	*g	2	4	5	1	1	1	3	1	3
520	3	1	6	1	1	6	1	71	1	9	1	2	1	11	5	1	25	1	6	67
540	2	9	6	1	5	2	15	1	2	48	2	7	1	3	1	4	21	1	1	2
560	1	27	3	26	2	1	1	2	5	7	3	*h	2	29	4	3	8	17	3	8
580	2	3	1	1	1	5	*i	1	3	4	1	4	1	1	13	1	34	1	2	7

Table I (*continued*)

$\sqrt[3]{2}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
600	1	3	3	7	1	3	1	1	4	2	69	1	3	12	34	1	2	**j*	1	**k*
620	4	1	1	12	3	4	2	3	1	1	1	1	1	2	1	1	6	16	1	2
640	27	2	13	4	1	1	1	3	11	1	1	3	1	53	2	15	1	1	1	1
660	1	1	2	2	1	1	3	3	1	9	1	1	10	3	1	1	2	1	2	2
680	1	10	9	1	2	5	1	2	2	1	1	2	4	7	1	5	1	1	1	1
700	4	2	25	16	5	4	1	3	2	3	13	1	49	6	2	5	1	1	2	7
720	3	2	1	1	1	4	1	1	1	5	1	2	1	2	1	1	1	1	2	2
740	4	1	2	1	10	5	4	8	10	2	4	1	1	1	4	1	41	1	3	1
760	56	3	1	1	3	1	3	1	5	6	6	3	1	2	1	1	1	12	1	10
780	2	1	1	1	1	50	5	1	2	6	5	1	2	5	6	5	2	77	1	4
800	2	1	1	1	1	1	4	2	1	2	1	1	1	1	1	6	2	1	1	7
820	1	5	1	1	1	1	2	2	1	1	5	2	1	5	1	1	1	4	1	2
840	17	1	20	7	4	2	1	1	1	2	1	4	7	3	4	3	3	5	31	1
860	1	2	2	6	1	1	1	1	1	1	1	1	6	1	6	1	1	23	20	1
880	22	16	4	2	1	3	2	1	1	2	5	5	1	1	15	3	1	1	2	1
900	1	1	4	2	1	2	23	6	10	3	2	3	6	2	1	1	1	1	1	1
920	4	3	2	1	2	1	4	10	7	1	1	1	1	3	3	2	**m*	1	1	11
940	2	6	1	4	1	2	2	9	3	1	1	3	22	4	1	93	1	3	1	4
960	2	1	2	3	2	1	2	11	1	1	3	1	2	1	28	23	4	11	1	9
980	1	4	3	1	6	1	2	1	12	2	6	19	1	4	4	2	1	1	1	1

$a = 534$ $b = 121$ $c = 186$ $d = 372$ $e = 186$ $f = 220$

$g = 255$ $h = 7451$ $i = 113$ $j = 151$ $k = 4941$ $m = 108$

Table II

$\sqrt[3]{2}$

Frequency Counts

1	422	22	4	56	2
2	165	23	3	67	1
3	91	25	3	69	2
4	66	26	2	71	1
5	40	27	4	72	1
6	37	28	1	77	1
7	20	29	2	84	1
8	16	30	1	89	1
9	15	31	1	93	1
10	15	33	1	108	1
11	8	34	2	113	1
12	9	36	1	121	1
13	8	38	2	151	1
14	4	39	2	186	2
15	5	41	2	220	1
16	3	44	1	255	1
17	2	47	1	372	1
18	3	48	1	534	1
19	3	49	2	4941	1
20	4	50	1	7451	1
21	3	53	1		

Table III

$\sqrt[3]{2}$

n	a_n	q_n	r_n
36	534	3.06 E 19	13.844
42	121	8.95 E 22	2.530
73	69	4.48 E 39	0.799
89	89	3.61 E 48	0.835
92	186	1.56 E 52	1.618
115	372	6.69 E 65	2.560
269	72	1.90 E 146	0.219
376	186	4.78 E 194	0.421
421	220	4.51 E 216	0.447
468	84	1.19 E 239	0.154
494	56	1.01 E 253	0.098
511	255	3.59 E 260	0.430
528	71	3.38 E 268	0.117
540	67	1.32 E 276	0.106
572	7451	8.64 E 297	11.005
587	113	4.07 E 308	0.160
611	69	7.64 E 320	0.095
618	151	5.97 E 326	0.202
620	4941	2.97 E 330	6.568
654	53	1.50 E 347	0.068
761	56	5.13 E 395	0.063
786	50	7.54 E 406	0.054
798	77	4.14 E 414	0.082
937	108	3.81 E 475	0.099
956	93	1.37 E 486	0.084

Table I

$\sqrt[3]{3}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	2	3	1	4	1	5	1	1	6	2	5	8	3	3	4	2	6	4	4
20	1	3	2	3	4	1	4	9	1	8	4	3	1	3	2	6	1	6	1	3
40	1	1	1	1	12	3	1	3	1	1	4	1	6	1	5	1	2	1	3	3
60	11	8	1	*a	8	2	8	5	1	2	2	2	2	3	1	1	2	1	1	1
80	52	2	46	2	2	3	3	1	5	1	6	1	6	1	1	1	6	1	20	10
100	4	8	1	1	1	2	1	2	*b	1	2	5	6	1	4	3	1	1	*c	2
120	3	3	1	6	1	3	1	2	1	24	2	1	5	4	1	1	2	1	1	1
140	3	1	1	20	5	60	1	1	37	6	3	28	1	1	2	1	31	1	3	3
160	1	2	58	1	11	1	31	1	8	1	11	2	2	3	1	4	1	4	37	1
180	2	1	2	82	1	2	6	11	5	1	1	1	7	54	2	27	1	2	1	24
200	3	3	1	2	1	1	*d	1	14	2	89	1	2	4	1	5	2	1	2	2
220	3	2	1	5	1	4	2	1	15	2	2	3	10	2	1	1	1	1	1	1
240	9	2	67	1	1	1	9	5	2	1	3	1	60	1	3	1	1	3	1	9
260	2	18	1	3	1	1	3	3	2	1	3	1	5	4	2	1	1	3	1	6
280	2	39	1	3	4	1	2	1	1	2	1	1	4	3	1	1	4	2	3	1
300	4	1	1	1	1	1	5	*e	1	84	3	1	2	1	3	7	3	1	1	1
320	8	1	7	1	1	1	11	1	1	2	1	1	5	1	1	1	3	1	1	2
340	2	1	7	24	5	4	1	1	1	2	2	1	1	95	1	3	1	2	1	6
360	2	1	1	6	1	1	1	3	1	16	*f	6	4	1	9	4	3	*g	1	2
380	3	4	2	1	1	*h	2	6	4	2	1	1	5	2	4	1	1	2	1	7
400	6	1	2	10	*i	3	1	2	1	1	34	1	1	2	2	1	1	10	4	15
420	1	2	1	1	2	4	1	3	1	7	1	42	1	3	1	2	4	6	2	1
440	2	1	28	3	1	5	3	1	1	1	3	2	4	4	11	1	1	3	2	10
460	6	2	1	1	26	3	2	1	1	1	1	2	26	2	3	1	66	6	1	8
480	2	1	4	1	10	3	1	1	2	1	1	1	24	4	1	2	23	4	8	1
500	41	1	4	1	1	25	1	4	1	4	6	23	1	5	2	23	1	4	3	1
520	1	5	16	1	8	2	1	11	2	2	1	1	10	3	58	4	1	34	1	1
540	1	19	1	1	1	1	1	1	1	1	1	1	7	3	3	1	11	1	16	1
560	6	5	19	7	2	4	2	2	7	4	1	3	1	1	1	3	1	1	3	*j
580	8	1	1	10	6	2	8	23	5	2	1	17	2	2	15	1	1	1	20	1
600	3	6	1	1	3	1	1	1	4	2	26	1	2	12	2	8	1	15	2	3
620	54	3	2	22	1	1	3	1	2	1	92	1	1	1	4	1	3	4	4	1
640	1	1	1	1	12	2	1	18	1	5	9	1	1	5	3	1	1	1	9	2
660	1	4	1	2	1	12	1	1	15	1	1	1	3	1	6	2	2	2	12	21
680	1	3	1	15	3	4	4	6	1	10	1	1	1	1	1	5	2	4	25	2
700	3	1	2	2	2	3	9	71	14	2	1	1	1	2	4	1	1	2	*k	1
720	1	*m	1	1	1	1	1	6	3	1	*n	5	12	2	2	*p	4	9	1	1
740	1	3	1	1	3	7	1	1	5	1	*q	1	34	1	12	2	1	1	2	1
760	69	1	2	3	2	*r	1	1	4	1	2	1	1	10	1	5	3	1	1	1
780	1	4	2	1	2	3	1	3	10	21	1	1	1	1	2	1	13	1	10	2

Table I (*continued*)

$\sqrt[3]{3}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
800	1	4	21	39	1	19	1	1	5	1	38	1	2	1	28	56	5	3	1	1
820	2	11	2	2	1	1	1	1	5	2	11	5	3	1	2	1	6	4	1	4
840	1	3	18	1	1	1	1	6	3	1	5	1	2	1	1	1	54	1	16	2
860	1	1	20	1	1	1	9	2	2	21	1	12	2	2	1	1	1	1	4	1
880	1	5	2	1	1	1	2	3	4	1	13	2	1	8	1	*s	2	1	60	1
900	1	6	10	8	4	1	3	1	2	5	1	3	16	20	5	*t	1	1	7	8
920	1	4	1	1	2	7	4	1	33	1	1	1	12	1	11	3	2	1	11	2
940	41	5	1	1	2	17	1	1	1	2	7	2	2	2	2	1	1	1	1	1
960	15	2	28	2	3	1	3	13	4	1	1	1	3	72	1	13	8	1	2	1
980	2	2	3	1	6	1	1	3	1	*u	1	2	4	4	3	15	7	2	39	2

$a = 139$ $b = 249$ $c = 612$ $d = 220$ $e = 123$ $f = 131$
$g = 196$ $h = 729$ $i = 164$ $j = 396$ $k = 343$ $m = 137$
$n = 139$ $p = 268$ $q = 247$ $r = 1232$ $s = 116$ $t = 3502$
$u = 164$

Table II

$\sqrt[3]{3}$

Frequency Counts

1	425	13	4	25	2	46	1	84	1	247	1
2	159	14	2	26	3	52	1	89	1	249	1
3	97	15	8	27	1	54	3	92	1	268	1
4	64	16	5	28	4	56	1	95	1	343	1
5	37	17	2	31	2	58	2	116	1	396	1
6	32	18	3	33	1	60	3	123	1	612	1
7	14	19	3	34	3	66	1	131	1	729	1
8	18	20	5	37	2	67	1	137	1	1232	1
9	10	21	4	38	1	69	1	139	2	3502	1
10	13	22	1	39	3	71	1	164	2		
11	12	23	4	41	2	72	1	196	1		
12	9	24	4	42	1	82	1	220	1		

Table III

$\sqrt[3]{3}$

n	a_n	q_n	r_n
64	139	6.85 E 30	2.118
81	52	9.42 E 38	0.614
109	249	3.42 E 54	2.077
119	612	8.70 E 60	4.575
146	60	7.46 E 73	0.363
163	58	1.94 E 84	0.307
184	82	2.18 E 96	0.379
194	54	3.33 E 102	0.233
207	220	3.57 E 110	0.885
211	89	9.94 E 113	0.347
243	67	3.00 E 128	0.231
253	60	4.08 E 133	0.200
308	123	1.53 E 157	0.345
310	84	1.31 E 159	0.235
354	95	5.96 E 177	0.236
371	131	9.09 E 185	0.310
378	196	7.22 E 191	0.450
386	729	1.24 E 197	1.631
405	164	4.22 E 207	0.347
477	66	2.34 E 242	0.121
535	58	2.72 E 274	0.093
580	396	9.29 E 296	0.585
621	54	5.56 E 320	0.074
631	92	2.84 E 326	0.124
708	71	2.10 E 363	0.085
719	343	5.43 E 369	0.406
722	137	1.49 E 372	0.161
731	139	4.61 E 376	0.162
736	268	3.91 E 381	0.307
751	247	3.47 E 389	0.278
761	69	2.10 E 395	0.077
766	1232	6.02 E 399	1.349
816	56	2.01 E 425	0.057
857	54	6.13 E 443	0.054
896	116	1.77 E 462	0.110
899	60	3.23 E 464	0.057
916	3502	3.38 E 477	3.209
974	72	1.22 E 507	0.062
990	164	1.91 E 515	0.139

Table I

	$\sqrt[3]{4}$																			
	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	1	1	2	2	1	3	2	3	1	3	1	30	1	4	1	2	9	6	4
20	1	1	2	7	2	3	2	1	6	1	1	1	25	1	7	7	1	1	1	1
40	**a*	1	3	2	1	3	60	1	5	1	8	5	6	1	4	20	1	4	1	1
60	14	1	4	4	1	1	1	1	7	3	1	1	2	1	3	1	4	4	1	1
80	1	3	1	34	8	2	10	6	3	1	2	31	1	1	1	4	3	44	1	45
100	93	12	1	7	1	1	5	12	1	1	2	4	19	1	12	1	16	1	8	1
120	1	2	1	**b*	1	1	1	6	3	1	6	1	2	2	2	3	2	6	1	5
140	20	1	2	1	78	2	1	12	2	2	4	22	2	11	4	6	23	99	1	12
160	4	4	1	1	2	7	2	1	4	1	1	2	1	2	1	9	7	1	2	4
180	1	1	1	1	10	2	56	11	2	1	7	1	2	1	4	1	1	9	1	4
200	4	9	1	2	1	4	1	17	1	1	4	26	4	1	1	1	12	1	11	3
220	1	20	10	1	4	2	5	3	5	1	2	1	1	9	3	1	8	1	6	3
240	13	1	3	5	6	5	1	1	18	1	1	3	3	8	1	3	1	12	1	2
260	8	2	8	3	1	2	44	11	5	7	1	35	1	1	2	1	1	4	2	1
280	1	1	1	5	1	1	1	2	4	6	1	3	17	2	18	1	3	1	1	1
300	3	1	1	5	1	3	1	4	3	3	2	2	6	2	3	9	15	78	1	2
320	1	1	1	3	1	3	1	2	1	1	20	1	1	1	5	1	2	3	5	8
340	1	1	1	6	12	2	1	4	1	11	2	3	1	1	1	6	5	6	5	1
360	3	1	1	1	4	3	2	1	1	1	4	1	5	10	2	3	2	1	**c*	1
380	5	2	1	23	2	9	1	2	2	4	2	3	1	1	2	1	3	1	37	1
400	1	1	2	79	2	4	10	1	2	4	3	7	3	2	5	1	2	1	3	**d*
420	2	1	1	8	1	1	1	1	2	2	1	2	6	1	2	2	2	4	15	1
440	2	3	1	8	24	2	2	1	1	1	2	1	16	7	5	3	7	7	3	16
460	1	1	1	1	1	1	41	1	3	1	2	5	4	1	41	1	1	2	3	1
480	1	6	29	1	14	3	1	2	2	3	1	3	1	2	28	2	1	1	2	27
500	1	2	1	4	1	3	4	**e*	1	8	2	1	4	1	1	2	1	1	1	1
520	2	3	3	1	2	1	**f*	1	4	2	1	2	5	1	1	2	2	12	1	13
540	33	1	2	1	4	13	1	2	4	7	1	5	24	4	3	1	8	1	1	1
560	1	10	3	2	55	1	1	1	12	1	2	3	1	10	3	1	1	1	**g*	1
580	58	2	6	4	34	1	1	1	3	1	2	1	1	3	11	56	1	7	2	2
600	2	3	1	6	2	17	2	1	15	1	1	6	3	1	8	9	1	**h*	1	1
620	24	17	2	1	**i*	1	**j*	9	25	1	1	1	1	1	2	1	1	3	4	2
640	3	3	33	2	1	13	4	6	1	1	1	1	4	1	1	23	8	1	26	4
660	7	1	4	4	2	2	3	1	1	1	1	2	4	1	3	5	7	6	2	2
680	21	4	1	5	2	1	5	1	3	2	1	1	1	1	3	2	2	1	4	9
700	1	50	8	10	2	2	1	1	2	1	1	27	1	24	12	1	11	5	3	1
720	1	1	5	3	2	3	12	2	6	4	2	2	1	1	1	6	1	4	1	1
740	2	8	4	20	1	9	3	2	2	20	1	8	1	27	1	1	1	3	1	1
760	2	1	1	11	3	12	1	1	6	3	6	2	5	5	4	1	24	1	1	2
780	2	1	12	2	1	5	2	1	1	2	1	1	2	4	38	1	9	1	3	4

Table I (*continued*)

$\sqrt[3]{4}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
800	1	1	1	2	6	4	13	1	3	1	3	2	2	1	4	5	1	3	1	2
820	5	1	2	3	10	2	1	8	2	10	14	2	5	3	1	2	2	14	1	1
840	1	1	1	1	1	6	2	1	1	15	3	2	2	1	2	1	4	4	3	3
860	2	3	3	1	11	41	1	10	1	1	7	1	1	1	1	2	7	1	3	2
880	1	2	11	31	1	1	3	1	3	9	1	2	1	46	3	20	1	1	2	1
900	1	12	1	3	4	9	1	1	2	6	1	1	1	1	4	1	1	3	3	3
920	1	1	1	1	4	54	3	1	5	4	3	2	2	2	1	4	4	1	1	1
940	3	1	1	44	2	2	46	1	8	1	1	1	2	5	1	1	2	5	5	1
960	3	1	1	6	1	13	1	1	11	8	5	1	20	1	1	1	1	1	2	3
980	2	1	2	6	4	3	39	1	1	1	1	1	1	2	4	3	5	6	2	10

$a = 266$ $\quad b = 745$ $\quad c = 372$ $\quad d = 110$ $\quad e = 511$ $\quad f = 144$
$g = 14902$ $\quad h = 139$ $\quad i = 303$ $\quad j = 2470$

Table II

$\sqrt[3]{4}$

Frequency Counts

1	412	22	1	54	1
2	164	23	3	55	1
3	100	24	5	56	2
4	69	25	2	58	1
5	39	26	2	60	1
6	32	27	3	78	2
7	19	28	1	79	1
8	20	29	1	93	1
9	14	30	1	99	1
10	12	31	2	110	1
11	11	33	2	139	1
12	15	34	2	144	1
13	6	35	1	266	1
14	4	37	1	303	1
15	4	38	1	372	1
16	3	39	1	511	1
17	4	41	3	745	1
18	2	44	3	2470	1
19	1	45	1	14902	1
20	8	46	2		
21	1	50	1		

Table III

$\sqrt[3]{4}$

n	a_n	q_n	r_n
41	266	1.93 E 19	6.868
47	60	5.59 E 22	1.248
101	93	9.85 E 51	0.808
124	745	8.43 E 65	5.136
145	78	2.00 E 76	0.460
158	99	5.29 E 86	0.508
187	56	1.53 E 100	0.249
318	78	2.14 E 167	0.205
379	372	6.02 E 194	0.842
404	79	1.26 E 207	0.168
420	110	2.84 E 216	0.223
508	511	4.53 E 260	0.861
527	144	4.29 E 268	0.236
565	55	1.14 E 289	0.084
579	14902	1.09 E 298	22.025
581	58	6.43 E 299	0.086
596	56	2.54 E 308	0.079
618	139	9.69 E 320	0.191
625	303	7.55 E 326	0.407
627	2470	1.87 E 330	3.283
702	50	2.38 E 367	0.060
926	54	2.40 E 475	0.050

Table I

$\sqrt[3]{5}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	1	2	2	4	3	3	1	5	1	1	4	10	17	1	14	1	1	**a*	1
20	1	1	1	1	1	2	2	1	3	2	1	13	5	1	1	1	13	2	41	1
40	4	12	1	5	2	7	1	1	3	33	2	1	1	1	1	1	1	3	2	2
60	1	15	12	8	10	48	1	2	1	1	3	4	1	**b*	1	13	2	4	1	1
80	49	3	10	1	8	1	1	1	1	4	1	60	7	2	2	2	3	3	2	1
100	3	2	1	61	1	10	2	8	3	4	4	2	33	2	1	1	1	7	3	4
120	2	1	1	1	42	1	6	1	3	1	2	1	3	2	1	1	1	1	1	1
140	2	1	2	3	2	2	1	3	1	1	3	12	2	45	3	7	31	1	6	5
160	2	1	1	3	1	1	5	2	3	1	1	4	1	1	1	4	1	6	1	1
180	2	3	2	1	8	1	7	1	35	2	8	6	1	4	2	1	3	1	2	1
200	12	1	7	20	1	2	42	1	1	2	5	1	3	1	81	2	21	1	5	1
220	1	4	8	9	1	52	1	2	1	2	2	5	1	13	2	3	1	1	1	9
240	2	1	3	13	2	17	2	2	10	1	11	1	2	1	2	9	2	1	1	1
260	2	11	1	5	2	1	9	1	3	1	17	1	6	1	1	1	51	2	**c*	2
280	8	1	1	1	1	1	25	1	5	6	3	1	2	2	3	4	1	4	9	4

Table I (*continued*)

	$\sqrt[3]{5}$																			
	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
300	1	1	2	1	1	1	1	1	1	1	2	4	1	2	5	1	2	11	1	1
320	1	1	1	2	1	3	3	2	1	21	1	2	1	2	1	1	38	1	4	2
340	1	5	19	4	1	1	1	2	1	2	1	1	2	4	1	2	1	3	1	1
360	2	1	1	1	4	48	5	1	6	1	1	**d*	1	3	**e*	23	1	51	12	2
380	35	1	1	2	3	2	1	59	1	1	2	2	3	1	1	2	3	5	2	1
400	2	5	2	1	3	**f*	1	5	2	1	4	3	2	5	61	2	1	5	1	2
420	1	5	2	1	2	1	3	6	3	2	9	8	1	1	8	13	1	1	2	1
440	5	1	1	21	9	1	2	1	1	3	1	1	2	2	4	1	**g*	1	1	2
460	1	6	1	50	3	2	1	4	1	1	20	5	4	2	1	1	1	1	1	1
480	7	9	10	2	9	1	**h*	1	12	2	5	2	2	1	**i*	1	6	1	5	1
500	9	1	4	1	23	1	3	15	3	1	2	2	2	10	2	1	25	3	2	55
520	1	1	2	1	3	2	1	7	2	2	5	1	18	3	1	2	8	3	1	3
540	1	1	5	5	1	1	1	1	4	1	1	21	1	8	1	3	2	1	5	1
560	4	3	1	2	1	13	1	1	1	2	2	1	13	2	1	5	72	1	1	4
580	1	2	1	1	1	10	2	1	3	1	3	3	1	4	2	2	1	6	6	1
600	1	1	1	1	1	29	2	2	3	1	1	3	2	1	1	1	1	2	6	1
620	1	3	1	28	20	3	1	6	2	2	1	6	1	29	1	2	15	5	3	3
640	1	5	1	1	2	1	2	2	5	3	1	1	3	2	6	7	2	1	1	4
660	4	3	1	1	3	1	3	1	3	1	37	1	8	15	1	2	4	1	10	9
680	1	2	3	1	4	1	3	7	1	1	**j*	1	1	1	3	1	4	1	1	2
700	5	9	3	8	2	2	5	2	6	4	1	3	7	2	2	1	4	1	8	1
720	2	1	5	1	1	1	2	1	1	8	4	4	2	13	3	1	7	1	45	2
740	2	1	5	3	3	2	1	2	5	13	1	2	1	7	10	1	3	1	1	1
760	1	75	7	1	63	1	2	1	3	5	1	1	5	1	1	1	3	1	1	3
780	1	1	5	2	2	1	1	1	1	16	2	1	3	1	1	3	1	1	1	9
800	1	37	6	1	1	1	40	2	**k*	1	5	1	**m*	7	4	1	1	1	7	2
820	1	2	2	2	2	4	1	1	2	2	4	1	5	2	8	2	3	5	1	5
840	3	7	5	1	1	1	53	4	12	1	3	2	7	8	2	9	2	47	1	1
860	2	6	3	1	37	2	1	1	1	1	**n*	13	1	6	2	1	2	3	2	1
880	1	3	4	1	1	4	1	2	1	1	3	1	1	2	1	2	2	3	1	1
900	3	1	6	1	7	1	1	1	4	2	1	4	1	1	2	23	1	1	1	1
920	5	1	3	1	10	16	**p*	1	6	1	9	1	1	2	1	13	1	8	2	1
940	3	8	23	4	2	1	2	9	1	11	1	1	3	1	2	4	1	6	2	1
960	1	13	14	1	12	2	1	6	1	53	1	3	3	5	1	1	1	1	2	1
980	1	6	1	1	25	2	1	1	1	34	4	1	10	1	40	1	3	1	3	1

$a = 3052$ $b = 474$ $c = 854$ $d = 131$ $e = 170$ $f = 1051$
$g = 182$ $h = 326$ $i = 135$ $j = 13977$ $k = 451$ $m = 49968$
$n = 739$ $p = 121$

Table II

$\sqrt[3]{5}$

Frequency Counts

1	433	17	3	41	1	75	1
2	180	18	1	42	2	81	1
3	95	19	1	45	2	121	1
4	50	20	3	47	1	131	1
5	46	21	4	48	2	135	1
6	25	23	4	49	1	170	1
7	19	25	3	50	1	182	1
8	19	28	1	51	2	326	1
9	16	29	2	52	1	451	1
10	12	31	1	53	2	474	1
11	4	33	2	55	1	739	1
12	8	34	1	59	1	854	1
13	13	35	2	60	1	1051	1
14	2	37	3	61	2	3052	1
15	4	38	1	63	1	13977	1
16	2	40	2	72	1	49968	1

Table III

$\sqrt[3]{5}$

n	a_n	q_n	r_n
19	3052	4.28 E 11	162.730
74	474	1.23 E 40	5.511
92	60	9.82 E 49	0.548
104	61	2.43 E 56	0.491
215	81	1.88 E 109	0.331
226	52	2.20 E 116	0.200
277	51	3.15 E 140	0.162
279	854	5.44 E 143	2.636
372	131	1.57 E 183	0.315
375	170	1.07 E 186	0.402
378	51	1.34 E 189	0.120
388	59	4.04 E 195	0.134
406	1051	1.00 E 205	2.260
415	61	2.21 E 210	0.127
457	182	2.35 E 230	0.348
464	50	6.48 E 233	0.095
487	326	5.54 E 246	0.581
495	135	8.42 E 251	0.236
520	55	9.33 E 265	0.091
577	72	2.06 E 291	0.108
691	13977	2.20 E 345	17.792
762	75	1.46 E 379	0.087
765	63	7.47 E 381	0.073
809	451	4.93 E 401	0.491
813	49968	1.73 E 407	53.913
847	53	3.15 E 423	0.055
871	739	8.92 E 438	0.737
927	121	7.42 E 462	0.114
970	53	9.37 E 484	0.048

Table I

$\sqrt[3]{7}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	1	10	2	16	2	1	4	2	1	21	1	3	5	1	2	1	1	2	11
20	5	1	3	1	2	27	4	1	*a	8	1	2	1	1	3	1	3	2	6	4
40	1	2	1	5	1	1	2	1	1	1	3	2	8	1	2	2	4	5	1	1
60	36	1	1	1	1	2	1	2	31	2	1	1	7	1	1	1	1	6	7	6
80	5	7	1	6	1	6	9	6	*b	5	33	8	2	1	1	1	6	2	1	1
100	1	1	2	13	7	1	1	1	17	3	2	1	1	1	23	*c	1	*d	1	72
120	1	4	6	6	1	11	6	12	7	3	5	1	4	6	2	2	1	2	4	4
140	1	1	9	11	6	2	1	15	6	2	1	1	2	1	3	5	1	2	1	1
160	1	1	1	15	1	35	1	2	1	2	4	2	1	11	2	1	1	1	1	15
180	4	1	1	2	22	2	1	31	2	2	1	1	3	1	10	2	1	2	1	1
200	2	1	1	8	1	5	1	1	6	2	1	14	1	4	7	5	2	6	6	1
220	4	1	1	3	2	10	1	3	2	16	2	34	1	1	1	18	3	1	7	1
240	3	12	1	1	15	3	2	1	2	2	1	2	2	17	3	2	2	3	5	8
260	1	56	1	2	1	25	1	3	1	3	3	8	20	8	1	3	1	1	1	8
280	1	4	12	1	3	6	3	1	3	7	19	3	1	1	13	1	1	1	1	10
300	2	2	1	1	3	15	1	4	1	*e	1	7	1	8	2	1	1	2	13	3
320	7	56	3	2	4	2	4	2	5	1	1	9	3	1	1	7	4	1	6	1
340	47	1	2	31	6	2	4	1	4	7	9	1	1	3	9	2	1	14	1	3
360	1	5	5	1	6	15	1	5	1	*f	6	1	1	11	4	1	6	8	19	1
380	1	1	8	2	6	4	19	3	14	14	3	1	11	1	6	1	1	2	2	1
400	1	1	1	5	2	1	1	1	1	4	2	3	2	1	1	1	1	1	2	3
420	1	2	1	4	1	2	12	1	1	13	3	1	1	2	*g	4	1	1	1	14
440	1	10	1	*h	7	60	9	32	1	6	2	13	1	1	1	2	1	98	1	1
460	1	2	1	2	1	*i	1	4	3	2	2	4	4	1	2	35	2	25	8	3
480	3	1	1	6	1	1	1	10	1	2	3	2	4	2	2	5	2	6	2	2
500	1	1	1	10	1	1	1	6	1	35	30	1	2	1	1	1	3	2	1	3
520	4	1	9	1	1	9	6	1	1	15	2	1	5	1	3	3	1	1	2	5
540	2	60	3	23	1	2	1	1	1	4	11	9	13	1	4	1	*j	2	1	11
560	3	2	1	1	5	2	52	1	4	1	8	2	3	2	3	25	1	14	1	1
580	1	7	8	1	2	11	15	1	2	14	2	1	6	1	2	7	1	1	2	2
600	1	1	1	2	1	1	1	1	1	9	*k	3	3	2	5	1	1	1	1	3
620	1	8	1	1	1	11	2	3	2	1	1	3	4	1	3	1	4	1	1	6
640	19	1	1	2	1	1	1	1	1	3	8	7	99	1	41	1	11	51	1	1
660	7	17	2	1	7	1	5	1	4	15	1	1	1	3	1	6	2	1	69	1
680	3	1	18	2	2	2	4	2	13	11	1	6	1	2	1	14	3	3	1	3
700	7	1	12	2	78	2	1	1	1	1	*m	5	1	10	5	2	1	2	5	1
720	6	3	2	2	10	13	4	27	1	2	1	1	44	19	4	3	1	3	1	1
740	2	2	5	3	1	1	1	1	1	1	1	2	1	1	1	11	1	8	6	1
760	7	1	1	2	19	1	5	3	1	1	3	1	3	1	1	44	1	34	1	5
780	3	8	1	2	3	*n	11	1	4	10	3	1	1	2	2	2	18	16	6	3

Table I (*continued*)

$\sqrt[3]{7}$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
800	63	1	1	5	1	19	24	1	1	7	53	1	1	4	2	1	1	**p*	2	1
820	5	2	7	5	4	1	1	3	1	30	1	1	1	2	1	7	1	7	1	1
840	24	2	1	4	1	1	3	2	1	3	21	1	1	1	73	1	4	2	1	1
860	4	9	1	2	4	2	3	1	1	3	7	2	11	2	1	4	8	8	15	11
880	18	17	1	1	2	7	43	2	3	1	1	6	2	1	10	1	5	3	3	3
900	8	22	4	3	1	4	4	4	6	1	7	1	1	9	1	18	3	4	1	1
920	2	1	2	7	1	2	2	1	15	1	1	2	5	1	2	52	3	1	87	1
940	3	1	1	1	6	10	1	1	1	3	2	4	1	6	1	2	1	3	2	3
960	1	1	1	5	2	1	1	8	1	1	8	1	2	4	1	**q*	2	8	7	2
980	84	4	1	11	2	2	12	3	1	1	1	3	4	12	2	1	9	1	72	2

$a = 282$ $b = 104$ $c = 277$ $d = 429$ $e = 303$ $f = 341$
$g = 110$ $h = 197$ $i = 118$ $j = 133$ $k = 689$ $m = 115$
$n = 111$ $p = 202$ $q = 628$

Table II

$\sqrt[3]{7}$

Frequency Counts

1	409	18	5	43	1	104	1
2	161	19	7	44	2	110	1
3	88	20	1	47	1	111	1
4	55	21	2	51	1	115	1
5	34	22	2	52	2	118	1
6	40	23	2	53	1	133	1
7	29	24	2	56	2	197	1
8	24	25	3	60	2	202	1
9	13	27	2	63	1	277	1
10	12	30	2	69	1	282	1
11	17	31	3	72	2	303	1
12	7	32	1	73	1	341	1
13	8	33	1	78	1	429	1
14	8	34	2	84	1	628	1
15	11	35	3	87	1	689	1
16	3	36	1	98	1		
17	4	41	1	99	1		

Table III

$\sqrt[3]{7}$

n	a_n	q_n	r_n
29	282	6.14 E 15	9.209
89	104	2.74 E 44	1.066
116	277	7.19 E 59	2.096
118	429	3.10 E 62	3.120
120	72	2.27 E 64	0.507
262	56	1.28 E 135	0.185
310	303	4.11 E 160	0.834
322	56	7.57 E 167	0.147
370	341	2.07 E 195	0.770
435	110	2.82 E 226	0.214
444	197	1.46 E 232	0.374
446	60	6.16 E 234	0.112
458	98	4.21 E 242	0.178
466	118	2.06 E 246	0.211
542	60	2.86 E 283	0.093
557	133	1.21 E 293	0.200
567	52	4.71 E 298	0.077
611	689	1.20 E 321	0.940
653	99	6.74 E 339	0.127
658	51	1.79 E 344	0.065
679	69	1.36 E 355	0.086
705	78	9.95 E 369	0.093
711	115	1.50 E 373	0.135
786	111	4.29 E 411	0.118
801	63	9.92 E 421	0.065
811	53	5.06 E 428	0.054
818	202	5.07 E 432	0.204
855	73	2.51 E 450	0.071
936	52	3.44 E 494	0.046
939	87	1.21 E 497	0.077
976	628	2.69 E 513	0.535
981	84	5.88 E 517	0.071
999	72	1.14 E 528	0.060

Table I

$\alpha_1 = 2\cos\frac{2\pi}{7}$, Root of $x^3 + x^2 - 2x - 1$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	4	20	2	3	1	6	10	5	2	2	1	2	2	1	18	1	1	3	2
20	1	2	1	2	1	39	2	1	1	1	13	1	2	1	30	1	1	1	3	2
40	5	4	1	5	1	5	1	2	1	1	94	6	2	19	11	1	60	1	1	50
60	2	1	1	8	53	1	3	1	6	3	2	1	5	1	1	3	4	*a	1	2
80	1	3	3	7	9	1	2	10	3	1	22	1	*b	3	32	1	2	1	2	4
100	2	1	2	2	62	2	1	1	8	1	14	5	6	5	1	1	8	1	7	*c
120	1	1	2	2	1	2	2	2	2	30	3	1	13	1	19	3	1	4	1	1
140	2	1	33	1	10	1	13	2	26	1	1	1	9	1	9	1	6	13	1	5
160	1	1	1	6	1	9	1	7	1	3	1	1	1	12	1	1	4	3	1	8
180	1	2	2	2	1	2	4	1	1	1	1	3	1	19	1	3	3	2	2	1
200	13	3	4	1	1	1	3	1	1	1	2	1	3	19	1	3	9	3	2	4
220	1	3	1	6	1	25	20	1	1	2	1	2	5	1	1	1	14	1	13	1
240	1	1	55	8	1	1	24	17	1	11	17	1	4	1	1	9	1	1	1	3
260	1	1	1	1	20	4	1	45	1	2	4	1	1	2	1	1	2	11	*d	1
280	4	1	2	1	2	*e	1	3	5	3	7	3	25	2	3	1	1	2	1	1
300	1	1	5	1	1	5	2	1	1	1	1	1	1	3	2	1	4	2	1	5
320	1	1	6	1	30	1	62	1	36	11	7	1	21	1	19	1	15	1	2	12
340	5	5	9	2	1	5	1	1	1	2	6	1	3	1	1	3	*f	8	30	2
360	1	1	1	3	1	3	1	1	44	2	19	1	3	1	1	17	20	1	4	3
380	97	2	1	10	4	2	1	5	1	1	4	1	3	18	1	1	1	4	14	2
400	4	13	3	2	9	2	1	2	1	2	1	4	1	22	1	1	3	1	16	61
420	1	2	3	2	5	1	1	2	1	1	7	3	11	2	1	4	2	3	1	2
440	2	1	27	4	1	3	5	17	2	2	10	2	1	2	15	1	37	5	7	1
460	24	4	56	2	2	4	1	1	1	4	3	1	2	8	3	3	2	4	1	1
480	2	14	*g	1	16	1	5	1	1	1	1	*h	1	3	1	*i	1	1	13	2
500	2	3	2	1	1	1	2	1	11	1	8	4	2	72	3	1	5	7	1	3
520	7	2	1	4	4	2	9	2	1	2	3	10	1	1	2	1	13	1	5	1
540	4	4	2	1	1	6	3	1	3	1	1	5	2	1	49	1	10	2	1	11
560	1	3	4	1	6	1	2	1	2	1	2	2	1	13	1	3	2	2	3	2
580	1	2	*j	1	3	2	2	1	3	3	9	1	3	*k	3	2	3	12	6	11
600	3	2	1	1	1	2	2	2	1	4	1	3	4	7	1	1	3	1	1	1
620	7	1	16	19	9	1	6	8	1	75	1	2	3	7	3	1	1	2	4	2
640	2	4	2	1	1	27	1	1	1	9	15	1	9	3	12	1	13	3	2	16
660	1	1	18	2	3	1	6	1	7	4	1	1	1	1	2	2	4	4	1	3
680	8	1	19	3	21	1	3	1	4	3	3	1	2	2	1	*m	4	7	1	*n
700	1	1	2	1	6	1	2	1	1	3	4	1	2	2	2	22	3	1	9	4
720	3	1	5	1	1	3	5	20	1	12	2	1	1	1	1	2	87	2	2	2
740	59	1	1	2	1	4	17	3	1	1	1	1	2	3	2	1	1	2	3	3
760	1	*p	4	11	19	2	2	1	5	2	3	1	4	1	1	1	1	4	3	1
780	6	1	12	2	7	1	5	4	9	3	3	2	2	1	1	2	4	2	4	3

Table I (*continued*)

$$\alpha_1 = 2\cos\frac{2\pi}{7}, \quad \text{Root of } x^3 + x^2 - 2x - 1$$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
800	5	3	1	1	5	1	1	4	1	1	20	1	1	1	6	1	6	2	3	1
820	3	1	2	9	1	1	1	10	10	**q*	1	1	10	1	3	1	14	1	2	2
840	5	1	1	2	3	2	6	1	3	1	12	1	25	9	**r*	1	1	3	1	26
860	34	2	2	5	3	1	2	5	2	3	2	1	1	2	1	8	2	1	1	5
880	3	1	2	9	32	1	1	3	4	1	3	1	1	3	1	1	20	2	2	1
900	3	10	57	1	**s*	20	1	1	1	66	1	26	1	4	4	6	5	50	1	1
920	5	3	1	1	6	21	4	4	1	1	1	1	2	7	5	3	9	5	3	1
940	4	2	1	2	1	17	1	59	3	1	8	1	1	10	3	4	2	5	1	1
960	1	2	1	14	5	7	1	1	6	46	1	2	4	6	3	1	3	8	24	7
980	1	1	2	1	3	11	4	1	14	1	13	2	1	2	2	1	7	2	2	1

$a = 636$ $b = 119$ $c = 425$ $d = 202$ $e = 136$ $f = 699$
$g = 424$ $h = 165$ $i = 114$ $j = 283$ $k = 267$ $m = 716$
$n = 108$ $p = 704$ $q = 904$ $r = 124$ $s = 152$

Table II

$$\alpha_1 = 2\cos\frac{2\pi}{7}, \quad \text{Root of } x^3 + x^2 - 2x - 1$$

Frequency Counts

1	401	15	3	33	1	59	2	136	1
2	168	16	4	34	1	60	1	152	1
3	109	17	6	36	1	61	1	165	1
4	60	18	3	37	1	62	2	202	1
5	40	19	9	39	1	66	1	267	1
6	23	20	8	44	1	72	1	283	1
7	19	21	3	45	1	75	1	424	1
8	13	22	3	46	1	87	1	425	1
9	19	24	3	49	1	94	1	636	1
10	12	25	3	50	2	97	1	699	1
11	10	26	3	53	1	108	1	704	1
12	7	27	2	55	1	114	1	716	1
13	12	30	4	56	1	119	1	904	1
14	7	32	2	57	1	124	1		

Table III

$\alpha_1 = 2\cos\frac{2\pi}{7}$, Root of $x^3 + x^2 - 2x - 1$

n	a_n	q_n	r_n
51	94	1.34 E 24	1.854
57	60	2.49 E 29	0.958
60	50	2.53 E 31	0.739
65	53	5.83 E 34	0.698
78	636	2.51 E 42	6.978
93	119	3.20 E 51	1.054
105	62	1.96 E 58	0.480
120	425	1.61 E 68	2.815
243	55	7.83 E 124	0.196
279	202	4.61 E 144	0.617
286	136	3.97 E 148	0.404
327	62	3.30 E 166	0.166
357	699	1.99 E 185	1.665
381	97	1.44 E 200	0.213
420	61	2.64 E 220	0.121
463	56	4.85 E 244	0.101
483	424	6.15 E 255	0.728
492	165	6.02 E 260	0.278
496	114	3.47 E 263	0.191
514	72	4.03 E 272	0.116
583	283	3.59 E 305	0.406
594	267	1.45 E 312	0.375
630	75	4.20 E 331	0.100
696	716	2.26 E 366	0.856
700	108	8.11 E 369	0.129
737	87	2.72 E 387	0.098
741	59	1.95 E 390	0.066
762	704	2.68 E 400	0.770
830	904	5.77 E 434	0.909
855	124	2.41 E 448	0.121
903	57	3.80 E 472	0.053
905	152	5.92 E 474	0.141
910	66	2.45 E 478	0.061
918	50	2.34 E 484	0.045
948	59	1.04 E 500	0.052

Table I

α_2, Root of $x^5 - x - 1$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	1	5	1	42	1	3	24	2	2	1	16	1	11	1	1	2	31	1	12	5
20	1	7	11	1	4	1	4	2	2	3	4	2	1	1	11	1	41	12	1	8
40	1	1	1	1	1	9	2	1	5	4	1	25	4	6	11	1	4	1	6	1
60	1	1	2	2	2	4	11	1	4	1	3	2	8	1	3	3	6	21	11	2
80	1	1	10	2	1	3	2	8	1	10	4	3	1	1	1	1	2	1	1	18
100	7	4	1	2	2	6	1	1	2	2	6	20	1	43	3	2	4	2	1	1
120	2	3	1	4	4	1	3	1	1	12	1	2	2	3	2	14	1	5	1	7
140	1	2	2	10	2	2	2	1	38	1	59	2	1	4	1	4	2	4	23	13
160	1	1	1	3	1	32	2	1	3	6	4	1	4	1	1	1	1	91	1	7
180	2	8	1	18	2	2	1	1	28	2	1	12	1	3	4	55	1	1	2	2
200	6	1	3	1	16	1	2	1	1	**a*	1	1	4	3	2	1	2	1	4	1
220	1	2	43	3	3	1	55	1	1	2	10	1	2	1	12	12	36	1	8	1
240	18	1	3	1	1	1	6	1	24	2	1	1	1	6	1	3	1	1	11	1
260	4	2	1	1	1	2	2	1	73	1	1	4	**b*	54	1	8	4	2	3	4
280	1	1	1	1	99	4	2	2	1	4	1	1	1	10	1	1	1	1	1	1
300	1	2	1	5	3	21	5	1	6	2	1	3	2	1	1	4	1	1	1	85
320	8	1	1	30	1	3	2	3	1	1	3	1	1	1	85	6	1	1	3	2
340	1	10	1	1	**c*	3	1	1	17	1	12	1	1	3	5	2	1	3	1	3
360	1	5	1	1	1	**d*	1	2	1	2	1	9	3	1	1	3	1	4	3	7
380	1	69	5	1	2	2	3	1	1	8	17	13	1	2	3	2	11	1	8	2
400	4	5	1	2	1	5	1	1	1	5	2	1	10	1	1	1	1	1	6	3
420	3	1	5	2	2	6	4	1	10	1	2	1	1	1	4	2	1	1	2	1
440	4	1	2	4	5	1	1	3	1	5	6	1	4	1	**e*	3	18	2	11	9
460	9	2	20	1	10	2	4	1	1	1	5	3	2	2	2	4	3	1	1	8
480	1	7	4	1	3	12	16	1	1	2	2	2	2	3	5	2	1	3	1	16
500	2	1	1	2	4	1	3	5	2	12	1	1	1	12	1	2	26	21	7	2
520	1	2	8	2	2	1	1	1	2	2	1	1	1	3	1	1	1	1	39	4
540	1	29	18	1	8	13	3	1	1	1	1	1	8	1	4	1	3	2	2	2
560	1	5	2	5	1	5	2	8	8	2	8	5	1	4	3	2	2	2	3	3
580	7	2	4	4	2	18	6	**f*	6	32	5	13	2	3	6	1	5	2	1	1
600	1	1	1	3	12	2	1	1	2	1	1	48	1	1	1	13	1	5	1	4
620	1	1	5	1	1	3	1	2	1	21	2	2	3	12	1	3	1	1	3	2
640	3	1	**g*	11	5	1	1	12	2	2	2	2	3	3	14	1	42	17	1	1
660	1	2	1	2	1	1	5	2	8	1	2	18	2	27	1	14	1	1	3	1
680	1	4	2	3	3	3	1	2	1	9	1	1	1	1	4	1	17	4	3	12
700	1	25	15	5	1	2	2	6	1	7	7	5	1	5	1	7	1	2	1	1
720	1	**h*	1	1	1	2	2	1	1	34	4	5	4	16	3	4	1	1	1	10
740	46	2	1	1	1	5	1	1	2	1	7	10	1	3	2	1	1	2	1	7
760	4	**i*	1	4	6	1	4	1	1	8	1	1	15	2	3	16	7	1	6	1
780	3	1	1	1	7	1	1	1	1	3	4	1	1	10	1	1	4	2	1	47

Table I (*continued*)

α_2, Root of $x^5 - x - 1$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
800	1	3	3	6	1	**j*	74	14	2	24	22	**k*	3	9	2	5	3	2	4	2
820	1	19	2	1	2	8	4	2	5	7	4	1	1	4	3	25	1	1	**m*	**n*
840	2	44	3	1	50	1	1	2	1	10	1	15	1	1	**p*	5	2	1	3	37
860	5	7	1	4	4	1	26	2	2	**q*	1	7	2	1	11	2	1	2	1	1
880	1	5	1	6	1	1	4	14	3	1	2	1	1	12	1	9	52	1	9	6
900	2	2	4	1	33	3	3	**r*	1	1	23	7	1	2	9	1	7	1	2	1
920	1	1	7	4	1	1	1	17	9	2	3	1	14	35	1	1	1	6	9	12
940	1	4	2	**s*	1	5	3	1	1	1	2	5	3	7	1	32	8	1	6	1
960	1	3	**t*	1	25	1	1	1	26	3	1	3	1	1	7	2	17	6	1	1
980	6	4	1	6	2	23	1	3	2	5	8	1	3	10	1	30	1	13	1	2

$a = 761$ $b = 195$ $c = 166$ $d = 264$ $e = 701$ $f = 11644$
$g = 169$ $h = 673$ $i = 457$ $j = 409$ $k = 274$ $m = 174$
$n = 124$ $p = 172$ $q = 1033$ $r = 110$ $s = 684$ $t = 1292$

Table II

α_2, Root of $x^5 - x - 1$
Frequency Counts

1	406	16	6	31	1	48	1	169	1
2	162	17	6	32	3	50	1	172	1
3	89	18	7	33	1	52	1	174	1
4	67	19	1	34	1	54	1	195	1
5	41	20	2	35	1	55	2	264	1
6	28	21	4	36	1	59	1	274	1
7	23	22	1	37	1	69	1	409	1
8	21	23	3	38	1	73	1	457	1
9	11	24	3	39	1	74	1	673	1
10	14	25	4	41	1	85	2	684	1
11	11	26	3	42	2	91	1	701	1
12	16	27	1	43	2	99	1	761	1
13	6	28	1	44	1	110	1	1033	1
14	6	29	1	46	1	124	1	1292	1
15	3	30	2	47	1	166	1	11644	1

Table III

α_2, Root of $x^5 - x - 1$

n	a_n	q_n	r_n
151	59	2.97 E 77	0.344
178	91	3.13 E 91	0.444
196	55	1.29 E 102	0.239
210	761	5.27 E 109	3.096
227	55	1.58 E 118	0.208
269	73	2.00 E 138	0.235
273	195	3.54 E 141	0.609
274	54	1.91 E 143	0.166
285	99	1.35 E 149	0.294
320	85	8.07 E 163	0.230
335	85	3.32 E 171	0.219
345	166	1.87 E 177	0.413
366	264	4.47 E 187	0.621
382	69	1.92 E 195	0.157
455	701	1.44 E 229	1.347
588	11644	6.58 E 300	17.042
643	169	7.49 E 327	0.226
722	673	8.41 E 369	0.797
762	457	1.47 E 391	0.511
806	409	1.39 E 413	0.434
807	74	1.03 E 415	0.078
812	274	4.43 E 421	0.284
839	174	8.98 E 437	0.174
840	124	1.11 E 440	0.123
845	50	2.03 E 444	0.050
855	172	9.54 E 449	0.167
870	1033	3.09 E 461	0.979
897	52	3.27 E 474	0.048
908	110	2.20 E 482	0.100
944	684	3.68 E 503	0.594
963	1292	1.91 E 515	1.096

Table I

$$\alpha_3 = \sqrt[3]{2} + \sqrt{3}, \quad x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
0	2	1	*a	1	1	3	1	1	5	7	2	3	2	4	1	18	5	1	13	3
20	3	3	4	1	69	2	1	1	7	1	1	3	1	1	13	2	5	2	1	3
40	1	2	38	3	1	2	1	1	2	1	5	1	1	*b	1	1	6	1	2	5
60	1	1	9	4	1	5	2	1	4	5	1	1	18	3	3	2	24	3	1	1
80	1	2	74	3	2	4	3	1	1	10	1	1	1	1	4	1	1	1	3	7
100	8	*c	4	1	4	1	1	2	1	5	1	2	3	1	23	18	4	1	2	1
120	85	1	2	1	2	1	8	1	1	1	22	3	1	3	1	1	8	3	15	30
140	1	7	1	1	1	11	4	1	19	1	1	1	3	6	1	44	3	8	3	1
160	1	1	10	4	1	8	3	5	16	6	3	1	2	12	1	2	3	2	1	3
180	9	1	5	2	4	1	3	2	26	1	2	1	1	2	2	4	2	1	3	2
200	4	5	1	4	1	2	1	4	*d	1	1	8	4	1	1	9	1	1	2	5
220	1	4	2	2	1	29	4	1	*e	3	61	1	4	15	1	3	23	1	5	1
240	1	1	2	3	2	3	6	1	8	1	2	2	1	1	10	1	1	3	3	*f
260	1	1	1	1	1	3	7	8	1	42	1	3	1	1	2	1	2	6	2	1
280	1	1	1	7	16	1	1	1	1	3	37	1	7	38	63	3	1	14	6	1
300	1	4	1	1	2	4	1	6	1	1	1	1	3	1	30	4	1	4	1	8
320	10	4	3	*g	25	2	1	2	1	1	*h	1	1	1	2	2	6	2	9	1
340	13	1	2	4	4	1	1	19	1	1	3	2	3	1	2	1	1	1	4	1
360	7	2	1	*i	8	4	20	1	2	1	3	1	1	94	2	1	3	4	1	3
380	1	7	1	3	1	9	9	1	4	2	4	2	35	1	2	2	1	2	1	1
400	3	12	4	1	1	2	6	1	1	1	1	2	3	13	1	1	5	1	3	7
420	7	2	4	4	3	1	1	1	1	3	12	10	2	2	1	1	1	1	1	3
440	6	1	2	6	28	2	1	1	1	2	1	1	1	2	1	1	10	1	5	3
460	2	3	1	1	3	2	2	8	1	13	4	1	1	7	1	1	2	2	10	4
480	1	1	5	1	8	2	1	4	4	1	1	1	8	12	2	5	3	18	4	27
500	2	3	1	1	1	1	4	3	2	1	7	1	1	1	8	5	1	5	1	5
520	2	1	4	1	14	14	1	1	1	1	2	1	1	22	7	1	1	5	17	1
540	2	4	1	15	1	3	1	1	1	11	1	2	16	1	1	3	1	1	1	9
560	1	3	30	4	3	4	36	1	6	2	2	1	36	2	2	1	5	7	1	1
580	99	2	16	1	37	1	2	1	1	3	1	4	1	1	5	2	2	1	3	3
600	1	3	2	4	1	2	2	31	1	11	1	1	24	1	2	8	3	1	*j	1
620	*k	3	19	2	2	8	1	1	11	2	1	7	1	1	1	7	1	1	5	3
640	3	1	25	1	8	2	2	1	6	1	1	2	1	2	1	1	4	1	1	10
660	22	1	37	1	19	2	1	17	1	38	2	3	8	1	8	2	30	1	2	2
680	5	2	2	3	1	2	1	7	1	3	1	1	12	1	11	3	3	*m	6	1
700	30	1	1	2	1	1	7	1	5	7	1	75	1	12	1	2	1	1	7	1
720	1	2	2	1	1	7	1	1	3	1	1	27	10	4	1	6	2	1	*n	1
740	2	2	1	1	4	5	7	3	17	21	1	1	58	13	33	2	4	1	5	3
760	12	1	16	3	3	7	*p	1	1	13	2	*q	1	7	2	1	3	1	1	1
780	1	1	2	1	2	1	1	2	7	3	1	3	34	13	10	1	1	1	3	1

Table I (*continued*)

$\alpha_3 = \sqrt[3]{2} + \sqrt{3}, \quad x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$

	1	2	3	4	5	6	7	8	9	10	1	2	3	4	5	6	7	8	9	10
800	1	32	1	*r	3	55	3	2	1	6	1	3	1	2	2	1	2	5	1	7
820	1	34	1	5	1	13	1	2	8	1	9	5	1	21	3	2	4	1	1	1
840	1	1	1	5	7	1	1	2	1	2	3	5	28	1	1	11	1	4	1	3
860	2	47	2	3	14	1	1	2	29	1	1	1	7	1	3	1	3	1	2	1
880	8	1	1	1	2	1	4	2	2	2	1	1	2	1	2	13	2	1	50	13
900	23	1	2	5	6	1	2	1	2	1	53	1	6	3	3	3	23	1	1	1
920	5	1	11	1	4	5	*s	2	9	1	27	1	15	2	1	29	1	3	2	2
940	2	3	6	2	80	3	1	9	1	3	9	1	1	2	1	3	11	8	17	1
960	3	1	1	4	8	2	3	1	*t	6	1	1	2	1	*u	3	38	2	1	2
980	3	1	2	2	3	1	9	5	1	8	2	7	2	1	1	1	5	8	1	10

$a = 123$ $\quad b = 1446$ $\quad c = 126$ $\quad d = 121$ $\quad e = 154$ $\quad f = 452$
$g = 315$ $\quad h = 135$ $\quad i = 103$ $\quad j = 120$ $\quad k = 331$ $\quad m = 184$
$n = 133$ $\quad p = 430$ $\quad q = 298$ $\quad r = 150$ $\quad s = 208$ $\quad t = 186$
$u = 138$

Table II

$\alpha_3 = \sqrt[3]{2} + \sqrt{3}, \quad x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$

Frequency Counts

1	418	20	1	42	1	123	1
2	156	21	2	44	1	126	1
3	105	22	3	47	1	133	1
4	56	23	4	50	1	135	1
5	38	24	2	53	1	138	1
6	20	25	2	55	1	150	1
7	30	26	1	58	1	154	1
8	25	27	3	61	1	184	1
9	12	28	2	63	1	186	1
10	11	29	3	69	1	208	1
11	8	30	5	74	1	298	1
12	7	31	1	75	1	315	1
13	11	32	1	80	1	331	1
14	4	33	1	85	1	430	1
15	4	34	2	94	1	452	1
16	5	35	1	99	1	1446	1
17	4	36	2	103	1		
18	4	37	3	120	1		
19	4	38	4	121	1		

Table III

	$\alpha_3 = \sqrt[3]{2} + \sqrt{3}$,	$x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23$	
n	a_n	q_n	r_n
3	123	1.24 E 2	****
25	69	1.45 E 14	2.462
54	1446	3.93 E 28	24.700
83	74	2.19 E 43	0.779
102	126	1.22 E 53	1.074
121	85	1.36 E 63	0.608
209	121	8.04 E 107	0.497
229	154	3.52 E 118	0.578
231	61	6.49 E 120	0.224
260	452	9.82 E 135	1.473
295	63	1.06 E 154	0.180
324	315	6.59 E 169	0.818
331	135	4.31 E 174	0.341
364	103	4.51 E 189	0.240
374	94	1.00 E 196	0.212
581	99	2.74 E 294	0.148
619	120	6.52 E 313	0.168
621	331	2.18 E 316	0.459
698	184	9.56 E 356	0.226
712	75	7.91 E 364	0.091
739	133	9.57 E 377	0.154
753	58	3.86 E 386	0.066
767	430	2.47 E 398	0.472
772	298	4.13 E 402	0.324
804	150	1.49 E 418	0.158
806	55	2.48 E 420	0.057
899	50	2.45 E 463	0.048
911	53	6.77 E 470	0.050
927	208	5.44 E 480	0.189
945	80	9.75 E 491	0.071
969	186	4.09 E 505	0.161
975	138	2.61 E 509	0.119

The program to do the calculation was written in Fortran, using machine-language subroutines for multiple-precision integer arithmetic to handle the coefficients of the polynomials. The calculation of a_n was done in floating-point arithmetic (approximately 14 significant digits), using a floating-point approximation to $P_n(x)$ (suitably scaled). This procedure avoids the use of multiple-precision arithmetic in any trial-and-error steps, and so makes for greater efficiency. One could be even more efficient, using an idea suggested by Lehmer [5], and compute several successive partial quotients from an approximation to $P_n(x)$. It is possible

to find $P_{n+m}(x)$ from $P_n(x)$ and $a_n, a_{n+1}, \ldots, a_{n+m-1}$ with less multiple-precision calculation than is needed to find all the intervening polynomials explicitly. The additional complication in the program, however, did not seem worthwhile, since the results given here were obtained by the simpler method in about 6 minutes on an IBM 360/91. A listing of the actual program may be obtained on request from Trotter.

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Addendum to Continued Fractions for Some Algebraic Numbers

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References [2], [3], and [4] came to our attention after the proof-sheets of [1] had been corrected. It is clear that the computational method we used is essentially the same as that used in [2] and [4], and is presumably the same as that used in [3] (which does not give details of the computation).

Reference [4] gives the results of a χ^2-test comparing the observed frequencies of partial quotients of certain algebraic numbers with the theoretical frequencies for a "random" number. Results are reported for nine algebraic numbers, for each of which between 700 and 800 partial quotients were calculated. Nothing was found to suggest non-randomness except for a very low value of χ^2 (indicating unusually *good* agreement between expected and observed frequencies) for the expansion of the cube root of 2. After some discussion the authors remark "... the impression persists that the expansion of $2^{1/3}$ is peculiar. Probably the expansion will have to be carried to many more terms to verify or contradict this impression."

It therefore occurred to us that it might be worthwhile to exhibit the results of applying a similar χ^2-test to the expansions that we had calculated. Following [4], we divided the partial quotients into ten groups consisting of 1, 2, 3, 4, 5 and 6, 7 and 8, 9 through 12, 13 through 19, 20 through 40, and over 40. For each of the eight numbers for which we obtained expansions, we give the χ^2 value obtained from the distribution of the first 1000 partial quotients, and in the column headed P, the approximate probability that the χ^2 for a random sample would be no larger. (The probabilities are computed for the ordinary

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χ^2-distribution on nine degrees of freedom. This is not strictly correct because the partial quotients of a "random" number are not independent. The error involved is assessed in [4], and we agree with the authors that it is negligible for present purposes.) For the first six numbers (the numbers of degree 3) we give the same information for the distribution of the first 3000 partial quotients.

The rows of the table correspond to the numbers reported on in [1]. Thus the first five are the cube roots of 2, 3, 4, 5, and 7, and the last three are the positive roots of

$$x^3 + x^2 - 2x - 1, \quad x^5 - x - 1,$$

and

$$x^6 - 9x^4 - 4x^3 + 27x^2 - 36x - 23.$$

The results do not suggest any significant departure from random behavior. In particular the anomaly observed in [4] for the cube root of 2 appears not to persist when the expansion is carried further.

$N = 1000$		$N = 3000$	
χ^2	P	χ^2	P
4.61	0.13	5.59	0.22
8.41	0.51	10.33	0.68
8.47	0.51	7.71	0.44
8.07	0.47	9.48	0.61
10.22	0.67	13.32	0.85
8.08	0.48	7.72	0.44
4.08	0.09	—	—
12.73	0.83	—	—

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Index

The book gives an introduction to continued fractions and diophantine approximations, readable by undergraduates but also of interest at the research level because the theory leads immediately into unsolved problems. Emphasis is placed on classical numbers, and also phenomena valid for almost all numbers. For instance, the continued fraction for *e* is computed. Tables of computations done with W. Adams and H. Trotter have been added to the original edition to see experimental data concerning possible conjectures about the behavior of algebraic numbers with respect to their continued fractions and approximations by rational numbers. The subject is particularly interesting for undergraduates who can be put in contact with deep mathematics without a very extensive building of theories. One general idea is that algebraic numbers will exhibit a behavior that is the same as almost all numbers in a probabilistic sense, except under very specific structural conditions, namely quadratic numbers. Results for almost all numbers (due to Khintchine) show an interplay between calculus and number theory, which will also show undergraduates how analysis mixes with number theory.

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